Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 7

The Force Method of Analysis: An Introduction

Since twentieth century, indeterminate structures are being widely used for its obvious merits. It may be recalled that, in the case of indeterminate structures either the reactions or the internal forces cannot be determined from equations of statics alone. In such structures, the number of reactions or the number of internal forces exceeds the number of static equilibrium equations. In addition to equilibrium equations, compatibility equations are used to evaluate the unknown reactions and internal forces in statically indeterminate structure. In the analysis of indeterminate structure it is necessary to satisfy the equilibrium equations (implying that the structure is in equilibrium) compatibility equations (requirement if for assuring the continuity of the structure without any breaks) and force displacement equations (the way in which displacement are related to forces). We have two distinct method of analysis for statically indeterminate structure depending upon how the above equations are satisfied:

1. Force method of analysis (also known as flexibility method of analysis, method of consistent deformation, flexibility matrix method)

2. Displacement method of analysis (also known as stiffness matrix method).

In the force method of analysis, primary unknown are forces. In this method compatibility equations are written for displacement and rotations (which are calculated by force displacement equations). Solving these equations, redundant forces are calculated. Once the redundant forces are calculated, the remaining reactions are evaluated by equations of equilibrium.

In the displacement method of analysis, the primary unknowns are the displacements. In this method, first force -displacement relations are computed and subsequently equations are written satisfying the equilibrium conditions of the structure. After determining the unknown displacements, the other forces are calculated satisfying the compatibility conditions and force displacement relations. The displacement-based method is amenable to computer programming and hence the method is being widely used in the modern day structural analysis.

In general, the maximum deflection and the maximum stresses are small as compared to statically determinate structure. For example, consider two beams of identical cross section and span carrying uniformly distributed load as shown in Fig. 7.1a and Fig. 7.1b.

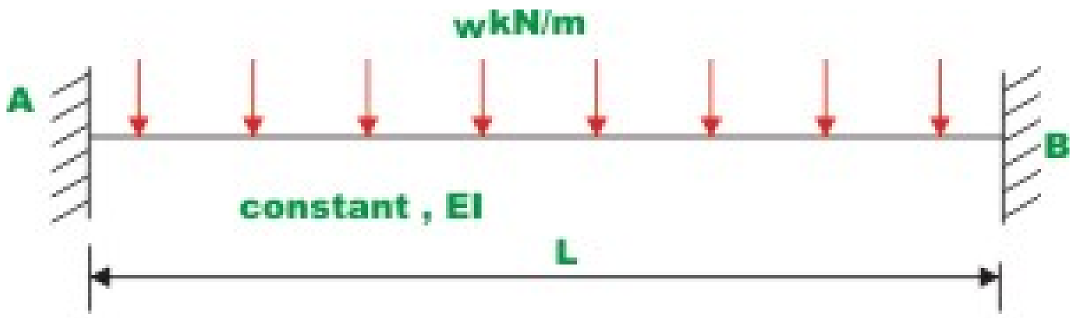


Fig. 7.1a Fixed - Fixed beam

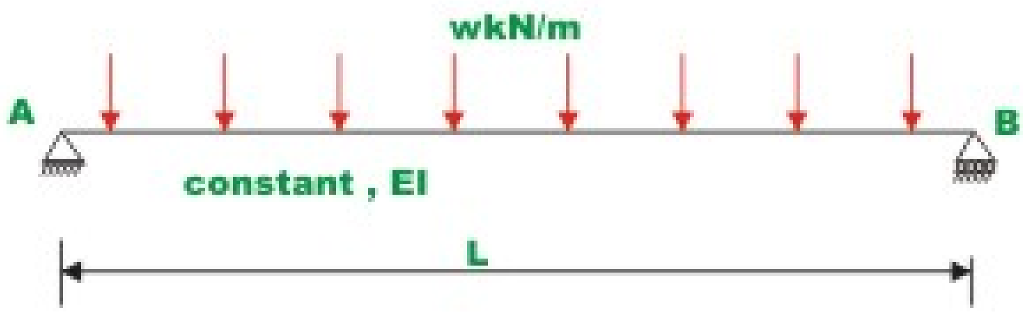


Fig. 7.1b Simply supported beam

The loads are also the same in both cases. In the first case, the beam is fixed at both ends and thus is statically indeterminate. The simply supported beam in Fig. 7.1b is a statically determinate structure. The maximum bending moment in case of fixed- fixed beam is $\frac{wL^2}{12}$ (which occurs at the supports) as compared to $\frac{wL^2}{8}$ (at the centre) in case of simply supported beam. Also in the present case, the deflection in the case of fixed- fixed beam $\left(\frac{wL^4}{384EI}\right)$ is five times smaller than that of simply supported beam $\left(\frac{5wL^4}{384EI}\right)$. Also, there is redistribution of stresses in the case of redundant structure. Hence if one member fails, structure does not collapse suddenly. The remaining members carry the load. The determinate structural system collapses if one member fails. However, there are disadvantages in using indeterminate structures. Due to support settlement,

there will be additional stresses in the case of redundant structures where as determinate structures are not affected by support settlement.
The analysis of indeterminate structure differs mainly in two aspects as compared to determinate structure.

a) To evaluate stresses in indeterminate structures, apart from sectional properties (area of cross section and moment of inertia), elastic properties are also required.
b) Stresses are developed in indeterminate structure due to support settlements, temperature change and fabrication errors etc.

Instructional Objectives

After reading this chapter the student will be
1. Able to analyse statically indeterminate structure of degree one.
2. Able to solve the problem by either treating reaction or moment as redundant.
3. Able to draw shear force and bending moment diagram for statically indeterminate beams.
4. Able to state advantages and limitations of force method of analysis.

7.1 Introduction.

In this lesson, a general introduction is given to the force method of analysis of indeterminate structure is given. In the next lesson, this method would be applied to statically indeterminate beams. Initially the method is introduced with the help of a simple problem and subsequently it is discussed in detail. The flexibility method of analysis or force method of analysis (or method of consistent deformation) was originally developed by J. Maxwell in 1864 and O. C. Mohr in 1874. Since flexibility method requires deflection of statically determinate structure, a table of formulas for deflections for various load cases and boundary conditions is also given in this lesson for ready use. The force method of analysis is not convenient for computer programming as the choice of redundant is not unique. Further, the bandwidth of the flexibility matrix in the force method is much larger than the stiffness method. However it is very useful for hand computation.

7.2 Simple Example

Consider a propped cantilever beam (of constant flexural rigidity EI, and span L), which is carrying uniformly distributed load of w kN/m., as shown in Fig. 7.2a. The beam is statically indeterminate i.e. its reaction cannot be evaluated from equations of statics alone. To solve the above problem by force method proceeds as follows.

1) Determine the degree of statical indeterminacy. In the present case it is one. Identify the reaction, which can be treated as redundant in the analysis. In the present case R_B or M_A can be treated as redundant. Selecting R_B as the redundant, the procedure is illustrated. Subsequently, it will be shown how to attack the problem by treating M_A as redundant.

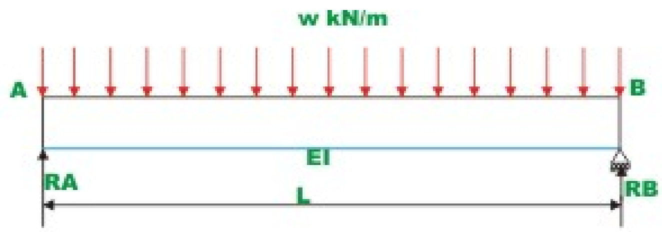


Fig. 7.2(a) Fixed - simply supported beam

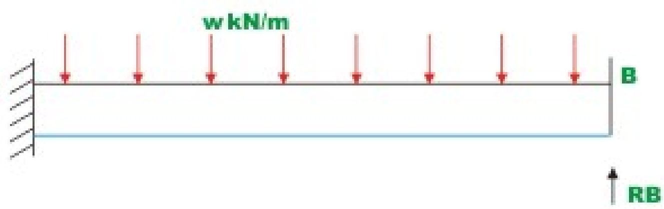


Fig. 7.2(b) Treating reaction R as redundant

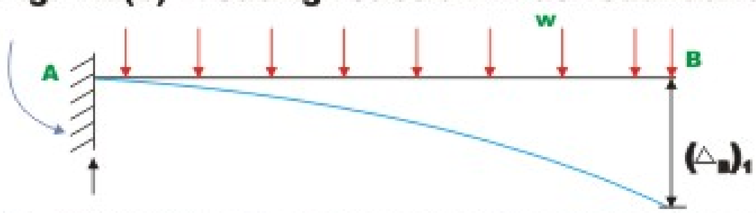


Fig. 7.2(c) Cantilever beam with external loading

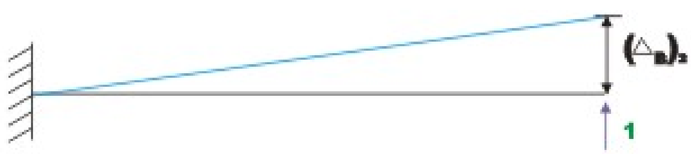
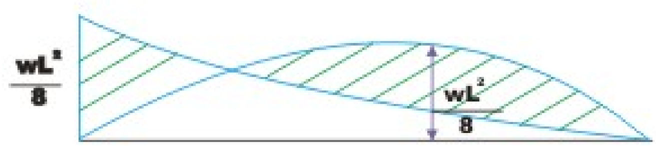
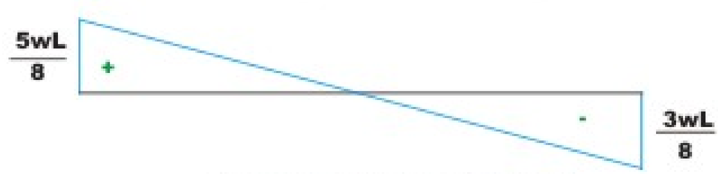


Fig. 7.2(d) Cantilever beam with a unit value of load along redundant R_B



Bending moment diagram



Shear force diagram

Fig. 7.2(e)

Solution with R_B as the redundant

2) After selecting R_B as redundant, express all other reactions in terms of the redundant R_B. This can be accomplished with the help of equilibrium equations. Thus,

$$R_A = wL - R_B \tag{7.1a}$$

and

$$M_A = \frac{wL^2}{2} - R_B L \tag{7.1b}$$

3) Now release the restraint corresponding to redundant reaction R_B. Releasing restraint in the present case amounts to removing the support at B. Now on the resulting cantilever beam (please note that the released structure is statically determinate structure), apply uniformly distributed load w and the redundant reaction R_B as shown in Fig. 7.2b. The released structure with the external loads is also sometimes referred as the primary structure.

4) The deflection at B of the released structure (cantilever beam, in the present case) due to uniformly distributed load and due to redundant reaction R_B could be easily computed from any one of the known methods (moment area method or unit load method). However it is easier to compute deflection at B due to uniformly distributed load and R_B in two steps. First, consider only uniformly distributed load and evaluate deflection at B, which is denoted by $(\Delta_B)_1$ as shown in Fig. 7.2c. Since R_B is redundant, calculate the deflection at B due to unit load at B acting in the direction of R_B and is denoted by $(\Delta_B)_2$ as shown in

In the present case the positive direction of redundant and deflections are assumed to act upwards. For the present case, $(\Delta_B)_1$ and $(\Delta_B)_2$ are given by,

$$(\Delta_B)_1 = -\frac{wL^4}{8EI} \tag{7.2a}$$

and

$$(\Delta_B)_2 = -\frac{L^3}{3EI} \tag{7.2b}$$

From the principle of superposition, the deflection at B, (Δ_B), is the sum of deflection due to uniformly distributed load $(\Delta_B)_1$ and deflection $R_B(\Delta_B)_2$ due to redundant R_B. Hence,

$$\Delta_B = (\Delta_B)_1 + R_B(\Delta_B)_2 \tag{7.2c}$$

5) It is observed that, in the original structure, the deflection at B is zero. Hence the compatibility equation can be written as,

$$\Delta_B = (\Delta_B)_1 + R_B(\Delta_B)_2 = 0 \tag{7.3a}$$

Solving the above equation, the redundant R_B can be evaluated as,

$$R_B = -\frac{(\Delta_B)_1}{(\Delta_B)_2} \tag{7.3b}$$

Substituting values of $(\Delta_B)_1$ and $(\Delta_B)_2$, the value of R_B is obtained as,

$$R_B = \frac{3wL}{8} \tag{7.3d}$$

The displacement at B due to unit load acting at B in the direction of R_B is known as the flexibility coefficient and is denoted in this course by a_{BB}.

6) Once R_B is evaluated, other reaction components can be easily determined from equations of statics. Thus,

$$M_A = \frac{wL^2}{8} \tag{7.4a}$$

$$R_A = wL - \frac{3wL}{8} = \frac{5wL}{8} \tag{7.4b}$$

7) Once the reaction components are determined, the bending moment and shear force at any cross section of the beam can be easily evaluated from equations of static equilibrium. For the present case, the bending moment and shear force diagram are shown in Fig. 7.2e.

Solution with M_A as redundant

1) As stated earlier, in the force method the choice of redundant is arbitrary. Hence, in the above problem instead of R_B one could choose M_A as the redundant reaction. In this section the above problem is solved by taking M_A as redundant reaction.

2) Now release (remove) the restraint corresponding to redundant reaction M_A. This can be done by replacing the fixed support at A by a pin. While releasing the structure, care must be taken to see that the released structure is stable and statically determinate.

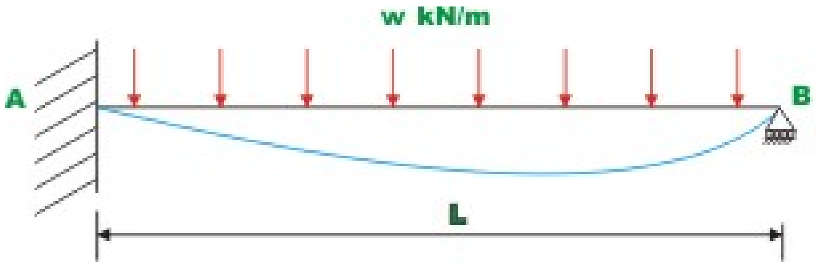


Fig.7.3(a) Actual structure

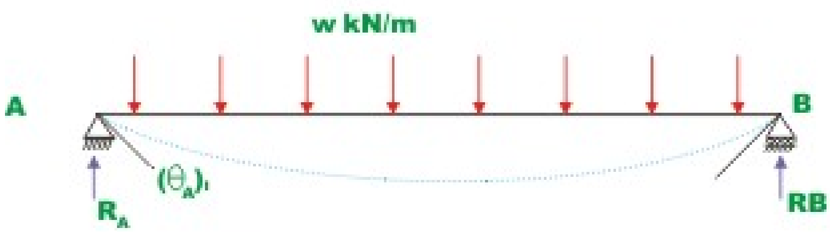


Fig. 7.3(b) Primary structure with external load load applied

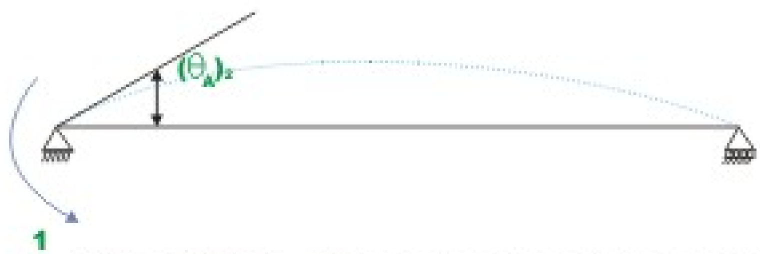


Fig. 7.3(c) Primary structure with unit moment applied in the direction of M_A 3)

Calculate the slope at A due to external loading and redundant moment M_A. This is done in two steps as shown in Fig. 7.3b and Fig.7.3c. First consider only uniformly distributed load (see Fig. 7.3b) and compute slope at A, i.e. $(\theta_A)_1$ from force displacement relations. Since M_A is redundant, calculate the slope at A due to unit moment acting at A in the direction of M_A which is denoted by $(\theta_A)_2$ as in Fig. 7.3c. Taking anticlockwise moment and anticlockwise rotations as positive, the slope at A, due to two different cases may be written as,

$$(\theta_A)_1 = -\frac{wL^3}{24EI} \tag{7.5a}$$

$$(\theta_A)_2 = \frac{L}{3EI} \tag{7.5b}$$

From the principle of superposition, the slope at A, θ_A is the sum of slopes $(\theta_A)_1$ due to external load and $M_A(\theta_A)_2$ due to redundant moment M_A. Hence

$$M_A = (\theta_A)_1 + M_A(\theta_A)_2 \tag{7.5c}$$

4) From the geometry of the original structure, it is seen that the slope at A is zero. Hence the required compatibility equation or geometric condition may be written as,

$$(\theta_A) = (\theta_A)_1 + M_A(\theta_A)_2 = 0 \tag{7.5d}$$

Solving for M_A,

$$M_A = -\frac{(\theta_A)_1}{(\theta_A)_2} \tag{7.5e}$$

Substituting the values of $(\theta_A)_1$, and $(\theta_A)_2$ in equation (7.5e), the value of M_A is calculated as

$$M_A = -\frac{-wL^3/24EI}{L/3EI}$$

$$M_A = \frac{wL^2}{8} \tag{7.5f}$$

5) Now other reaction components can be evaluated using equilibrium equations. Thus,

$$R_A = \frac{5wL}{8} \tag{7.6a}$$

$$R_B = \frac{3wL}{8} \tag{7.6b}$$

7.3 Summary

The force method of analysis may be summarized as follows.

Step 1. Determine the degree of statical indeterminacy of the structure. Identify the redundants that would be treated as unknowns in the analysis. Now, release the redundants one by one so that a statically determinate structure is obtained. Releasing the redundant reactions means removing constraint corresponding to that redundant reaction. As in the above propped cantilever beam, either reactions R_B or M_A can be treated as unknown redundant. By choosing R_B as the redundant, the propped cantilever beam can be converted into a cantilever beam (statically determinate) by releasing the roller support. Similarly by choosing moment as the redundant reaction, the indeterminate structure can be released into a determinate structure (i.e. a simply supported beam) by turning the fixed support into a hinged one. If the redundant force is an internal one, then releasing the structure amounts to introducing discontinuity in the corresponding member. The compatibility conditions for the redundant internal forces are the continuity conditions. That would be discussed further in subsequent lessons.

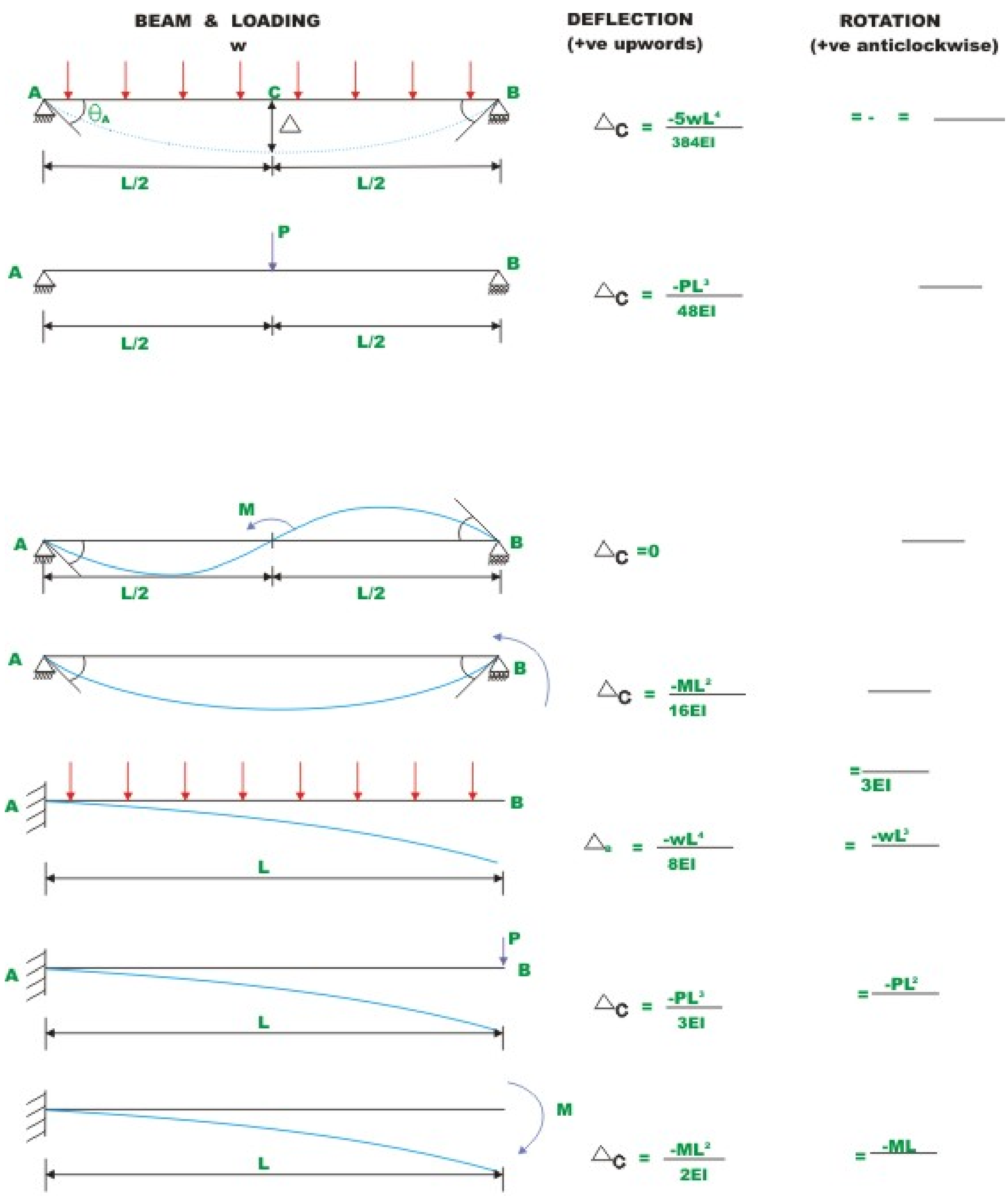
Step 2. In this step, calculate deflection corresponding to redundant action, separately due to applied loading and redundant forces from force displacement relations. Deflection due to redundant force cannot be evaluated without knowing the magnitude of the redundant force. Hence, apply a unit load in the direction of redundant force and determine the corresponding deflection. Since the method of superposition is valid, the deflections due to redundant force can be obtained by simply multiplying the unknown redundant with the deflection obtained from applying unit value of force.

Step 3. Now, calculate the total deflection due to applied loading and the redundant force by applying the principle of superposition. This computed total deflection along the redundant action must be compatible with the actual boundary conditions of the original structure. For example, if in the original structure, the deflection corresponding to the redundant reaction is zero then the total deflection must be equal to zero. If there is more than one redundant force then one could construct a set of equations with redundant forces as unknowns and flexibility coefficients as coefficients of the equations. The total number of equations equals the number of unknown redundants.

Step 4. In the last step, evaluate all other reactions and internal forces from the equilibrium equations.

The method of superposition or the force method as discussed above is applied to any type of structures, i.e. beams, truss and frames or combination of these structures. It is applicable for all general type of loadings.

The deflection of statically determinate structure can be obtained by unit-load method or by moment-area theorem or by any method known to the reader. However, the deflections of few prismatic beams with different boundary conditions and subjected to simple loadings are given in Fig. 7.4. These values will be of help in solving the problems of the present and subsequent lessons. However the students are strongly advised to practice deriving them instead of simply memorizing them.



Example 7.1

A continuous beam ABC is carrying a uniformly distributed load of 1 kN/m in addition to a concentrated load of 10 kN as shown in Fig.7.5a, Draw bending moment and shear force diagram. Assume EI to be constant for all members.

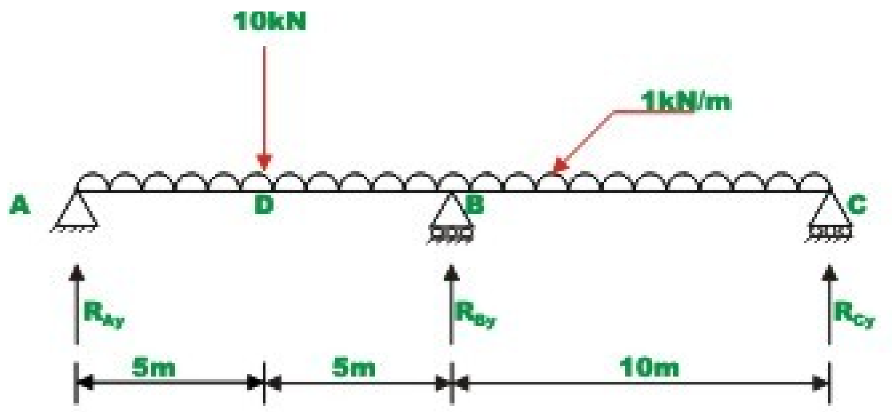


Fig. 7.5a Continuous beam

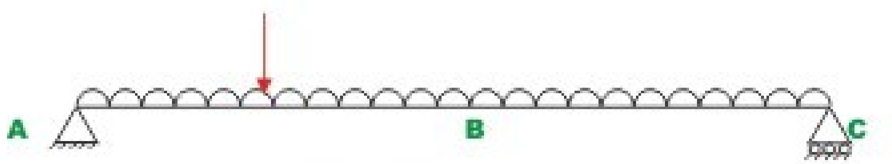


Fig. 7.5b Primary structure

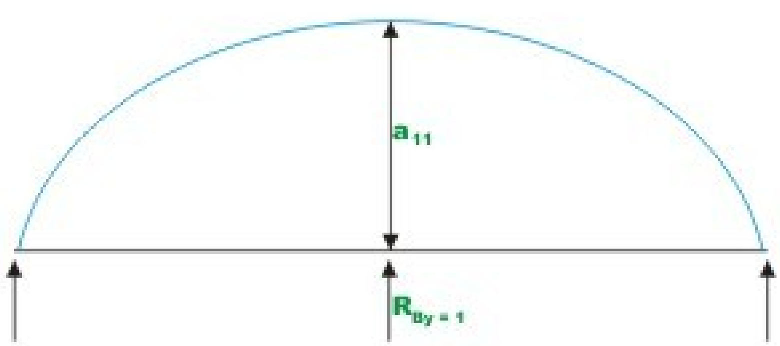


Fig. 7.5c Flexibility co-efficients

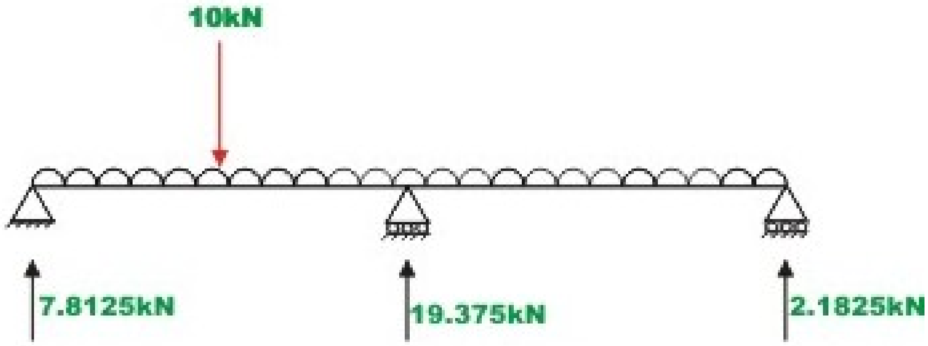


Fig. 7.5d Reactions

It is observed that the continuous beam is statically indeterminate to first degree. Choose the reaction at B, R_{By} as the redundant. The primary structure is a simply supported beam as shown in Fig.7.5b.
Now, compute the deflection at B, in the released structure due to uniformly distributed load and concentrated load. This is accomplished by unit load method. Thus,

$$\Delta_L = \frac{-2083.33}{EI} - \frac{1145.84}{EI}$$

$$\Delta_L = \frac{-3229.17}{EI} \quad (1)$$

In the next step, apply a unit load at B in the direction of R_{By} (upwards) and calculate the deflection at B of the following structure. Thus (see Fig. 7.5c),

$$a_{11} = \frac{L^3}{48EI} = \frac{166.67}{EI} \quad (2)$$

Now, deflection at B in the primary structure due to redundant R_B is,

$$\Delta_B = \frac{166.67}{EI} \times R_B \quad (3)$$

In the actual structure, the deflection at B is zero. Hence, the compatibility equation may be written as

$$\Delta_L + \Delta_B = 0 \tag{4}$$

Substituting for Δ_L and Δ_B in equation (4),

$$\frac{-3229.17}{EI} + \frac{166.67}{EI} R_B = 0 \tag{5}$$

Thus,

$$R_B = 19.375 \text{ kN}$$

The other two reactions are calculated by static equilibrium equations (vide Fig. 7.5d)

$$R_A = 7.8125 \text{ kN}$$
$$R_B = 2.8125 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 7.5e and Fig. 7.5f respectively.

Example 7.2

A propped cantilever beam AB is subjected to a concentrated load of 60 kN at 3m from end A as shown in Fig. 7.6a. Draw the bending moment and shear force diagrams by the force method. Assume that the flexural rigidity of the beam, EI to be constant throughout.

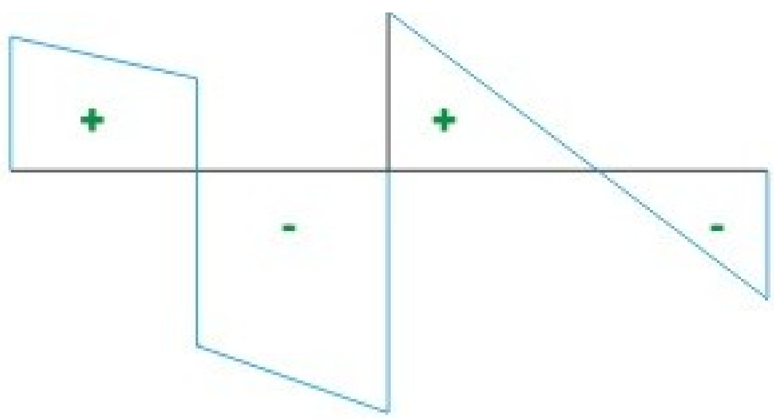


Fig. 7.5e Shear force diagram

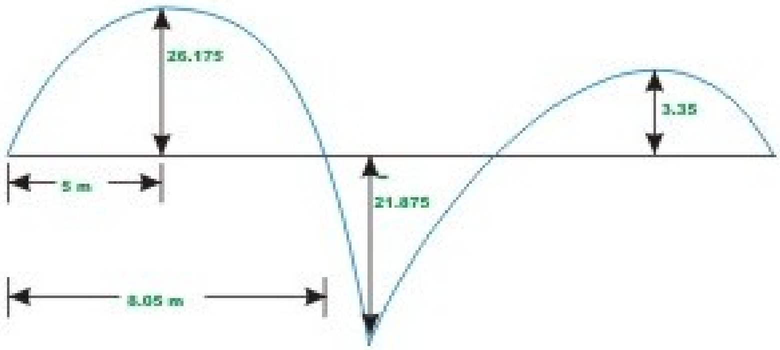


Fig. 7.5f BENDING MOMENT DIAGRAM.

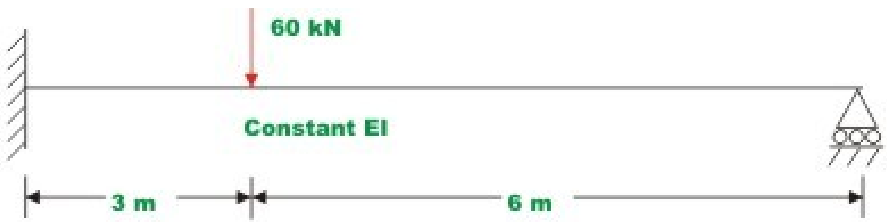
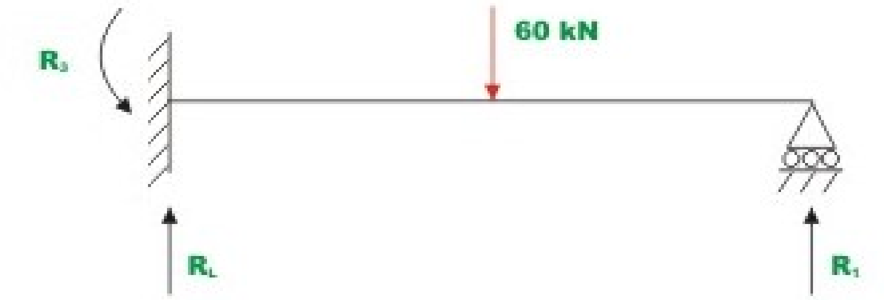


Fig. 7.6a Example 7.2

Fig. 7.6b Primary structure with external loading

Fig. 7.6c Primary structure with unit load applied along R_B



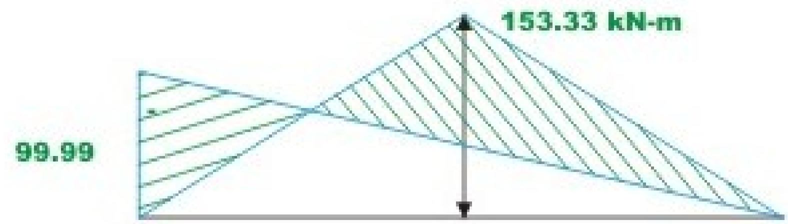


Fig.7.6d Bending moment diagram

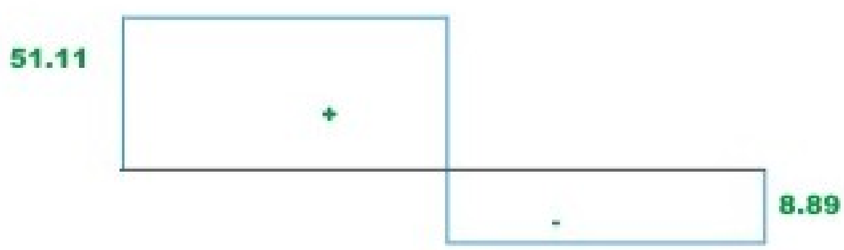


Fig.7.6e Shear force diagram

The given problem is statically indeterminate to first degree. Choose the reaction at B, R_1 as the redundant. After releasing the redundant, the determinate structure, a cantilever beam in this case is obtained. The cantilever beam with the applied loading is chosen in Fig 7.6b.

The deflection of the released structure is,

$$(\Delta_L)_1 = -\frac{60\times 3^3}{3EI} - \frac{60\times 3^2\times 6}{2EI}$$

$$(\Delta_L)_1 = \frac{-2160}{EI} \qquad (1)$$

The deflection at point B due to unit load applied in the direction of redundant R_1 is (vide Fig 7.6c)

$$a_{11} = \frac{9^3}{3EI} = \frac{243}{EI} \tag{2}$$

Now the deflection at B due to redundant R_1 is

$$(\Delta)_1 = \frac{243R_1}{EI} \tag{3}$$

From the original structure it is seen that the deflection at B is zero. Hence, the compatibility condition for the problem may be written as,

$$-\frac{2160}{EI} + \frac{243R_1}{EI} = 0 \tag{4}$$

Solving equation (4), the redundant R_1 is obtained.

$$R_1 = \frac{2160}{243} \\ = 8.89 \text{ kN} \tag{5}$$

The vertical reaction and fixed end moment at A can be determined from equations of statics. Thus,

$$R_2 = 51.11 \text{ kN}$$

$$R_3 = 99.99 \text{ kN.m} \tag{6}$$

Shear force and bending moment diagrams are shown in Fig. 7.6d and Fig. 7.6e respectively.

Summary

In this lesson flexibility matrix method or the method of consistent deformation or the force method of analysing statically indeterminate structures has been introduced with the help of simple problems. The advantages and limitations of flexibility matrix method have been discussed. Only simple indeterminate beam problem has been solved to illustrate the procedure. The principle of superposition has been used to solve statically indeterminate problems.