When the French mathematician Joseph Fourier (1768–1830) was trying to solve a problem in heat conduction, he needed to express a function f as an infinite series of sine and cosine functions:

$$\boxed{1} \qquad \begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &\qquad + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \end{aligned}$$

Earlier, Daniel Bernoulli and Leonard Euler had used such series while investigating problems concerning vibrating strings and astronomy.

The series in Equation 1 is called a *trigonometric series* or *Fourier series* and it turns out that expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series. In particular, astronomical phenomena are usually periodic, as are heartbeats, tides, and vibrating strings, so it makes sense to express them in terms of periodic functions.

We start by assuming that the trigonometric series converges and has a continuous function $f(x)$ as its sum on the interval $[-\pi, \pi]$, that is,

$$\boxed{2} \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad -\pi \leqslant x \leqslant \pi$$

Our aim is to find formulas for the coefficients a_n and b_n in terms of f. Recall that for a power series $f(x) = \Sigma\, c_n(x - a)^n$ we found a formula for the coefficients in terms of derivatives: $c_n = f^{(n)}(a)/n!$. Here we use integrals.

If we integrate both sides of Equation 2 and assume that it's permissible to integrate the series term-by-term, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{\pi} a_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, dx \\ &= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, dx \end{aligned}$$

But

$$\int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{n} \sin nx \bigg]_{-\pi}^{\pi} = \frac{1}{n} [\sin n\pi - \sin(-n\pi)] = 0$$

because n is an integer. Similarly, $\int_{-\pi}^{\pi} \sin nx\, dx = 0$. So

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0$$

and solving for a_0 gives

|||| Notice that a_0 is the average value of f over the interval $[-\pi, \pi]$.

3
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

To determine a_n for $n \geqslant 1$ we multiply both sides of Equation 2 by $\cos mx$ (where m is an integer and $m \geqslant 1$) and integrate term-by-term from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx\, dx$$

4
$$= a_0 \int_{-\pi}^{\pi} \cos mx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\, dx$$

We've seen that the first integral is 0. With the help of Formulas 81, 80, and 64 in the Table of Integrals, it's not hard to show that

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \qquad \text{for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases}$$

So the only nonzero term in (4) is $a_m \pi$ and we get

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = a_m \pi$$

Solving for a_m, and then replacing m by n, we have

5
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, 3, \ldots$$

Similarly, if we multiply both sides of Equation 2 by $\sin mx$ and integrate from $-\pi$ to π, we get

6
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, 3, \ldots$$

We have derived Formulas 3, 5, and 6 assuming f is a continuous function such that Equation 2 holds and for which the term-by-term integration is legitimate. But we can still consider the Fourier series of a wider class of functions: A **piecewise continuous function** on $[a, b]$ is continuous except perhaps for a finite number of removable or jump discontinuities. (In other words, the function has no infinite discontinuities. See Section 2.5 for a discussion of the different types of discontinuities.)

7 Definition Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the **Fourier series** of f is the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients a_n and b_n in this series are defined by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

and are called the **Fourier coefficients** of f.

Notice in Definition 7 that we are *not* saying $f(x)$ is equal to its Fourier series. Later we will discuss conditions under which that is actually true. For now we are just saying that *associated with* any piecewise continuous function f on $[-\pi, \pi]$ is a certain series called a Fourier series.

EXAMPLE 1 Find the Fourier coefficients and Fourier series of the **square-wave function** f defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

So f is periodic with period 2π and its graph is shown in Figure 1.

Engineers use the square-wave function in describing forces acting on a mechanical system and electromotive forces in an electric circuit (when a switch is turned on and off repeatedly). Strictly speaking, the graph of f is as shown in Figure 1(a), but it's often represented as in Figure 1(b), where you can see why it's called a square wave.

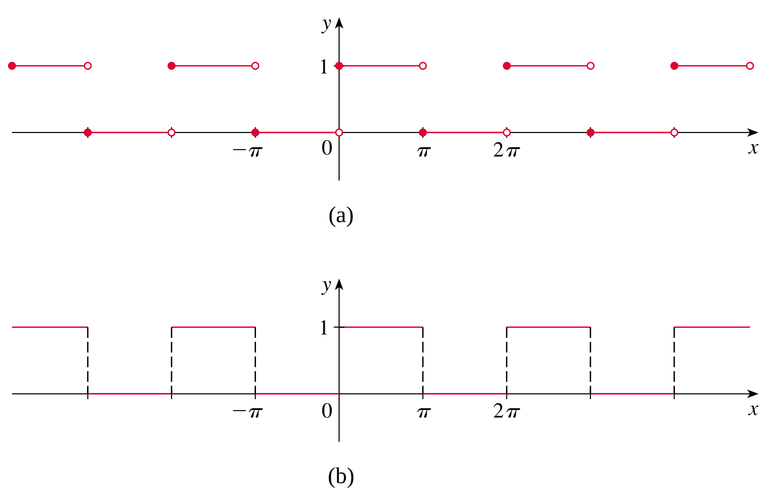


FIGURE 1
Square-wave function

SOLUTION Using the formulas for the Fourier coefficients in Definition 7, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{0} 0\, dx + \frac{1}{2\pi} \int_{0}^{\pi} 1\, dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

and, for $n \geqslant 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\frac{\sin nx}{n}\bigg]_0^{\pi} = \frac{1}{n\pi}(\sin n\pi - \sin 0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= -\frac{1}{\pi}\frac{\cos nx}{n}\bigg]_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - \cos 0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Note that $\cos n\pi$ equals 1 if n is even and -1 if n is odd.

The Fourier series of f is therefore

$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$$
$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

$$= \frac{1}{2} + 0 + 0 + 0 + \cdots$$
$$+ \frac{2}{\pi}\sin x + 0\sin 2x + \frac{2}{3\pi}\sin 3x + 0\sin 4x + \frac{2}{5\pi}\sin 5x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \frac{2}{7\pi}\sin 7x + \cdots$$

Since odd integers can be written as $n = 2k - 1$, where k is an integer, we can write the Fourier series in sigma notation as

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin(2k-1)x$$

In Example 1 we found the Fourier series of the square-wave function, but we don't know yet whether this function is equal to its Fourier series. Let's investigate this question graphically. Figure 2 shows the graphs of some of the partial sums

$$S_n(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots + \frac{2}{n\pi}\sin nx$$

when n is odd, together with the graph of the square-wave function.

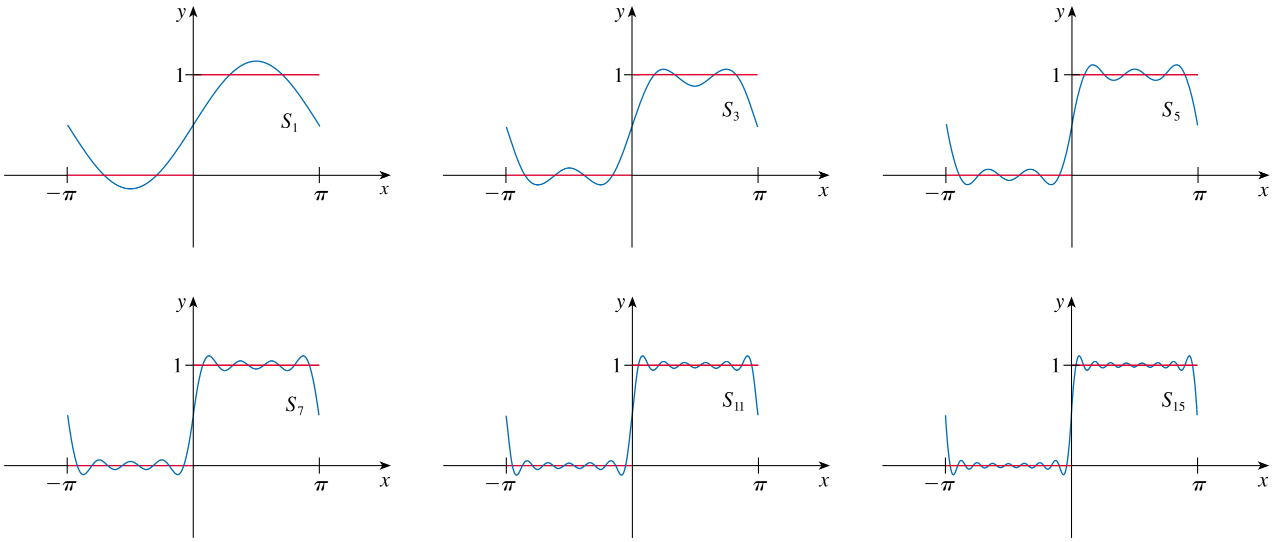


FIGURE 2 Partial sums of the Fourier series for the square-wave function

We see that, as n increases, $S_n(x)$ becomes a better approximation to the square-wave function. It appears that the graph of $S_n(x)$ is approaching the graph of $f(x)$, except where $x = 0$ or x is an integer multiple of π. In other words, it looks as if f is equal to the sum of its Fourier series except at the points where f is discontinuous.

The following theorem, which we state without proof, says that this is typical of the Fourier series of piecewise continuous functions. Recall that a piecewise continuous function has only a finite number of jump discontinuities on $[-\pi, \pi]$. At a number a where f has a jump discontinuity, the one-sided limits exist and we use the notation

$$f(a^+) = \lim_{x \to a^+} f(x) \qquad\qquad f(a^-) = \lim_{x \to a^-} f(x)$$

$\boxed{8}$ **Fourier Convergence Theorem** If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series (7) is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is

$$\tfrac{1}{2}[f(x^+) + f(x^-)]$$

If we apply the Fourier Convergence Theorem to the square-wave function f in Example 1, we get what we guessed from the graphs. Observe that

$$f(0^+) = \lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad f(0^-) = \lim_{x \to 0^-} f(x) = 0$$

and similarly for the other points at which f is discontinuous. The average of these left and right limits is $\frac{1}{2}$, so for any integer n the Fourier Convergence Theorem says that

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x = \begin{cases} f(x) & \text{if } n \neq n\pi \\ \frac{1}{2} & \text{if } x = n\pi \end{cases}$$

(Of course, this equation is obvious for $x = n\pi$.)

|||| Functions with Period 2L

If a function f has period other than 2π, we can find its Fourier series by making a change of variable. Suppose $f(x)$ has period $2L$, that is $f(x + 2L) = f(x)$ for all x. If we let $t = \pi x/L$ and

$$g(t) = f(x) = f(Lt/\pi)$$

then, as you can verify, g has period 2π and $x = \pm L$ corresponds to $t = \pm\pi$. The Fourier series of g is

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt\, dt \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt\, dt$$

If we now use the Substitution Rule with $x = Lt/\pi$, then $t = \pi x/L$, $dt = (\pi/L)\, dx$, and we have the following

9 If f is a piecewise continuous function on $[-L, L]$, its **Fourier series** is

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and, for $n \geqslant 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

|||| Notice that when $L = \pi$ these formulas are the same as those in (7).

Of course, the Fourier Convergence Theorem (8) is also valid for functions with period $2L$.

EXAMPLE 2 Find the Fourier series of the triangular wave function defined by $f(x) = |x|$ for $-1 \leqslant x \leqslant 1$ and $f(x + 2) = f(x)$ for all x. (The graph of f is shown in Figure 3.) For which values of x is $f(x)$ equal to the sum of its Fourier series?

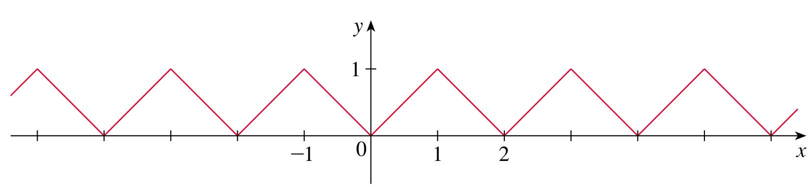


FIGURE 3
The triangular wave function

SOLUTION We find the Fourier coefficients by putting $L = 1$ in (9):

Notice that a_0 is more easily calculated as an area.

$$a_0 = \tfrac{1}{2}\int_{-1}^{1} |x|\, dx = \tfrac{1}{2}\int_{-1}^{0} (-x)\, dx + \tfrac{1}{2}\int_{0}^{1} x\, dx$$

$$= -\tfrac{1}{4}x^2\Big]_{-1}^{0} + \tfrac{1}{4}x^2\Big]_{0}^{1} = \tfrac{1}{2}$$

and for $n \geqslant 1$,

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x)\, dx = 2\int_{0}^{1} x\cos(n\pi x)\, dx$$

because $y = |x| \cos(n\pi x)$ is an even function. Here we integrate by parts with $u = x$ and $dv = \cos(n\pi x)\, dx$. Thus,

$$a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\, dx$$

$$= 0 - \frac{2}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

Since $y = |x| \sin(n\pi x)$ is an odd function, we see that

$$b_n = \int_{-1}^{1} |x| \sin(n\pi x)\, dx = 0$$

We could therefore write the series as

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos n\pi - 1)}{n^2\pi^2}\cos(n\pi x)$$

But $\cos n\pi = 1$ if n is even and $\cos n\pi = -1$ if n is odd, so

$$a_n = \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is

$$\frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x) - \frac{4}{25\pi^2}\cos(5\pi x) - \cdots$$

$$= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos((2k-1)\pi x)$$

The triangular wave function is continuous everywhere and so, according to the Fourier Convergence Theorem, we have

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos((2k-1)\pi x) \qquad \text{for all } x$$

In particular,

$$|x| = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos((2k-1)\pi x) \qquad \text{for } -1 \leqslant x \leqslant 1$$

Fourier Series and Music

One of the main uses of Fourier series is in solving some of the differential equations that arise in mathematical physics, such as the wave equation and the heat equation. (This is covered in more advanced courses.) Here we explain briefly how Fourier series play a role in the analysis and synthesis of musical sounds.

We hear a sound when our eardrums vibrate because of variations in air pressure. If a guitar string is plucked, or a bow is drawn across a violin string, or a piano string is struck, the string starts to vibrate. These vibrations are amplified and transmitted to the air. The resulting air pressure fluctuations arrive at our eardrums and are converted into electrical impulses that are processed by the brain. How is it, then, that we can distinguish between a note of a given pitch produced by two different musical instruments?

The graphs in Figure 4 show these fluctuations (deviations from average air pressure) for a flute and a violin playing the same sustained note D (294 vibrations per second) as functions of time. Such graphs are called **waveforms** and we see that the variations in air pressure are quite different from each other. In particular, the violin waveform is more complex than that of the flute.

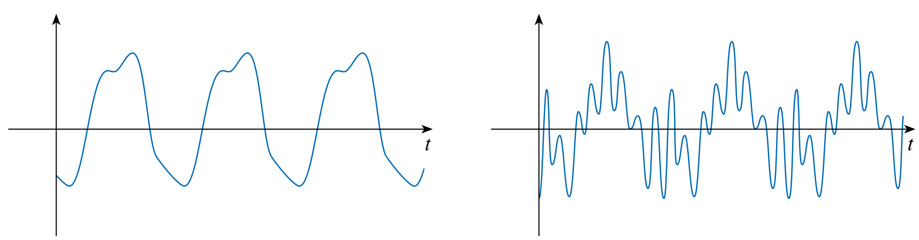


FIGURE 4
Waveforms

(a) Flute (b) Violin

We gain insight into the differences between waveforms if we express them as sums of Fourier series:

$$P(t) = a_0 + a_1 \cos\left(\frac{\pi t}{L}\right) + b_1 \sin\left(\frac{\pi t}{L}\right) + a_2 \cos\left(\frac{2\pi t}{L}\right) + b_2 \sin\left(\frac{2\pi t}{L}\right) + \cdots$$

In doing so, we are expressing the sound as a sum of simple pure sounds. The difference in sounds between two instruments can be attributed to the relative sizes of the Fourier coefficients of the respective waveforms.

The nth term of the Fourier series, that is,

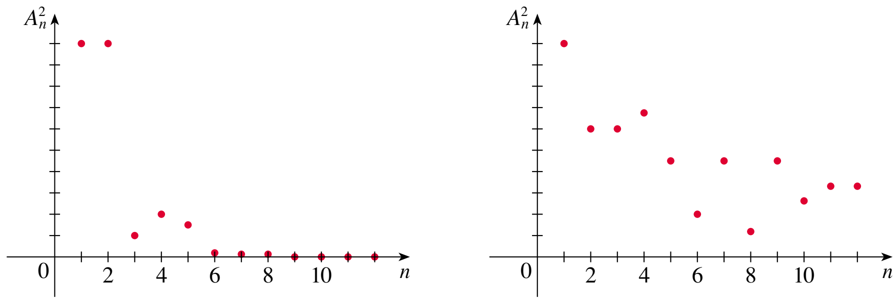
$$a_n \cos\left(\frac{n\pi t}{L}\right) + b_n\left(\frac{n\pi t}{L}\right)$$

is called the ***n*th harmonic** of P. The **amplitude** of the nth harmonic is

$$A_n = \sqrt{a_n^2 + b_n^2}$$

and its square, $A_n^2 = a_n^2 + b_n^2$, is sometimes called **energy** of the nth harmonic. (Notice that for a Fourier series with only sine terms, as in Example 1, the amplitude is $A_n = |b_n|$ and the energy is $A_n^2 = b_n^2$.) The graph of the sequence $\{A_n^2\}$ is called the **energy spectrum** of P and shows at a glance the relative sizes of the harmonics.

Figure 5 shows the energy spectra for the flute and violin waveforms in Figure 4. Notice that, for the flute, A_n^2 tends to diminish rapidly as n increases whereas, for the violin, the higher harmonics are fairly strong. This accounts for the relative simplicity of the flute waveform in Figure 4 and the fact that the flute produces relatively pure sounds when compared with the more complex violin tones.



(a) Flute (b) Violin

FIGURE 5
Energy spectra

In addition to analyzing the sounds of conventional musical instruments, Fourier series enable us to synthesize sounds. The idea behind music synthesizers is that we can combine various pure tones (harmonics) to create a richer sound through emphasizing certain harmonics by assigning larger Fourier coefficients (and therefore higher corresponding energies).

Exercises

1–6 A function f is given on the interval $[-\pi, \pi]$ and f is periodic with period 2π.

(a) Find the Fourier coefficients of f.

(b) Find the Fourier series of f. For what values of x is $f(x)$ equal to its Fourier series?

(c) Graph f and the partial sums S_2, S_4, and S_6 of the Fourier series.

1. $f(x) = \begin{cases} 1 & \text{if } -\pi \leqslant x < 0 \\ -1 & \text{if } 0 \leqslant x < \pi \end{cases}$

2. $f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ x & \text{if } 0 \leqslant x < \pi \end{cases}$

3. $f(x) = x$

4. $f(x) = x^2$

5. $f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ \cos x & \text{if } 0 \leqslant x < \pi \end{cases}$

6. $f(x) = \begin{cases} -1 & \text{if } -\pi \leqslant x < -\pi/2 \\ 1 & \text{if } -\pi/2 \leqslant x < 0 \\ 0 & \text{if } 0 \leqslant x < \pi \end{cases}$

7–11 Find the Fourier series of the function.

7. $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } 1 \leqslant |x| < 2 \end{cases} \qquad f(x+4) = f(x)$

8. $f(x) = \begin{cases} 0 & \text{if } -2 \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < 1 \\ 0 & \text{if } 1 \leqslant x < 2 \end{cases} \qquad f(x+4) = f(x)$

9. $f(x) = \begin{cases} -x & \text{if } -4 \leqslant x < 0 \\ 0 & \text{if } 0 \leqslant x < 4 \end{cases} \qquad f(x+8) = f(x)$

10. $f(x) = 1 - x, \quad -1 \leqslant x \leqslant 1 \qquad f(x+2) = f(x)$

11. $f(t) = \sin(3\pi t), \quad -1 \leqslant t \leqslant 1$

12. A voltage $E \sin \omega t$, where t represents time, is passed through a so-called half-wave rectifier that clips the negative part of the wave. Find the Fourier series of the resulting periodic function

$$f(t) = \begin{cases} 0 & \text{if } -\dfrac{\pi}{\omega} \leqslant t < 0 \\ E \sin \omega t & \text{if } 0 \leqslant t < \dfrac{\pi}{\omega} \end{cases} \qquad f(t + 2\pi/\omega) = f(t)$$

13–16 Sketch the graph of the sum of the Fourier series of f without actually calculating the Fourier series.

13. $f(x) = \begin{cases} -1 & \text{if } -4 \leqslant x < 0 \\ 3 & \text{if } 0 \leqslant x < 4 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } -1 \leqslant x < 0 \\ 1 - x & \text{if } 0 \leqslant x < 1 \end{cases}$

15. $f(x) = x^3, \quad -1 \leqslant x \leqslant 1$

16. $f(x) = e^x, \quad -2 \leqslant x \leqslant 2$

17. (a) Show that, if $-1 \leqslant x \leqslant 1$, then

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

(b) By substituting a specific value of x, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

18. Use the result of Example 2 to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

19. Use the result of Example 1 to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

20. Use the given graph of f and Simpson's Rule with $n = 8$ to estimate the Fourier coefficients a_0, a_1, a_2, b_1, and b_2. Then use them to graph the second partial sum of the Fourier series and compare with the graph of f.