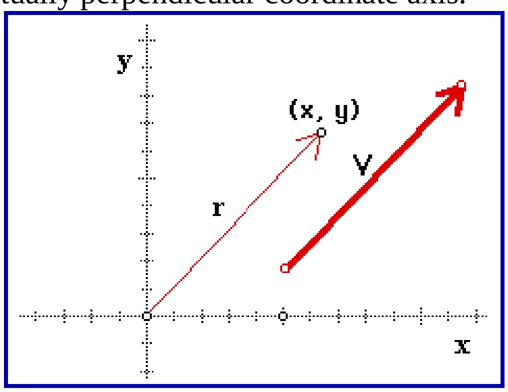
PHYS 3033 GENERAL RELATIVITY PART I
Chapter 3
The four dimensional space-time, four-vectors and four-tensors

"I have become imbued with great respect for mathematics, the subtler part of which I had, in my simplemindedness, regarded as pure luxury."
Albert Einstein (1916)

3.1 Mathematical introduction: vector spaces

In this lecture we shall introduce the concept of space-time as a four-dimensional vector space[1]. Since we live in a three-dimensional Euclidean space, we have a natural tendency to think of this as unique and special. But vectors and many concepts associated with them are not limited by the dimensionality of the space. In the following we denote the number of dimensions of a space by d. Then the hierarchy of the vector spaces is as follows[2]:

$d=2$ A vector is an ordered pair of numbers, $\vec{V}=(V_1,V_2)$. For example, the position vector can be written as $\vec{r}=(x,y)$, where x and y have their usual meanings as distances from the origin along mutually perpendicular coordinate axis.



We define the scalar product (or inner product) of the vectors as

$$\vec{r}\cdot\vec{r}\equiv xx+yy=x^2+y^2=r^2.$$

Hence the scalar product gives the length (absolute value) of a vector. **Since the length of a vector is a scalar quantity, it has the same value in any frame of reference**. Therefore the absolute value of a vector is invariant with respect to rotations and translations of the reference frame. r^2 is also known as the distance between two points.

$d=3$ A vector is an ordered triad or pair of numbers, $\vec{V}=(V_1,V_2,V_3)$. The position vector is $\vec{r}=(x,y,z)$, and its scalar product with itself is

$$\vec{r}\cdot\vec{r}=xx+yy+zz=x^2+y^2+z^2=r^2.$$

$d=4$ In four-dimensions a vector is an ordered set of four numbers, $\vec{V}=(V_1,V_2,V_3,V_4)$. The four-dimensional position vector is $\vec{s}=(x,y,z,u)$, where now we have four Cartesian coordinates, corresponding to the four rectangular axes. The scalar product is

$$\vec{s}\cdot\vec{s}=xx+yy+zz+uu=x^2+y^2+z^2+u^2=s^2. \tag{1}$$

$d = n$ A vector is an ordered set of n numbers, $\vec{V} = (V_1, V_2, V_3, V_4, ..., V_n)$, for any integer such that $2 \le n \le \infty$. The position vector may be written as $\vec{s} = (x_1, x_2, x_3, ..., x_n)$ and the inner product takes the form

$$\vec{s} \cdot \vec{s} = x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2 = s^2 . \qquad (2)$$

The vector spaces in which the distance between two points is described according to the prescription (2) are called Euclidian spaces. In a Euclidian space the distance is always well-defined and greater than zero, $s^2 \ge 0$.

3.2 The Minkowskian space-time continuum

In Newtonian physics, space and time are considered as separate entities and whether or not events are simultaneous is a matter that is regarded as obvious to any observer. In Einstein's concept of the physical universe, two observers in relative motion could disagree regarding the simultaneity of distant events.

As we have seen in the previous Chapter, in special relativity the quantity

$$s^2 = c^2 t^2 - x^2 - y^2 - z^2 \qquad (3)$$

is invariant with respect to the Lorentz transformations in the (ct, x) plane, and also with respect to translations and rotations in the (x, y, z) plane. Eq. (3) is very similar to Eq. (2), describing the distance between two points in a four-dimensional Euclidian vector space, with ct having the dimensions of length. This analogy led the German mathematician Hermann Minkowski (1864-1909) to unify space and time in a single framework, called space-time. **Space-time provides the true theater for every event in the lives of stars, atoms and people.**

H. Minkowski (1864-1909)

Hermann Minkowski taught at several universities, Bonn, Königsberg and Zurich. In Zurich, Einstein was a student in several of the courses he gave. By 1907 Minkowski realized that the work of Lorentz and Einstein could be best understood in a non-Euclidean space. He considered space and time, which were formerly thought to be independent, to be coupled together in a four dimensional 'space-time continuum'. Minkowski worked out a four-dimensional treatment of electrodynamics. The space-time continuum provided a framework for all later mathematical work in relativity. These ideas were used by Einstein in developing the general theory of relativity. As a mathematician Minkowski spent much of his time investigating quadratic forms and continued fractions.

As Minkowski wrote, ‘’**Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a union of the two will preserve an independent reality**”. Space is different for different observers. Time is different for different observers. Space-time is the same for everyone [3].

However, there is a major difference between the Minkowskian space-time (3) and the usual four-dimensional vector space (2). These differences arise due to the appearance of the minus signs in the definition of the interval (or distance) between two events. Initially, Minkowski introduced the fourth coordinate in the form $x_4 = ict$, which brings the interval to a Euclidian form. This representation is still used today, but in the following we shall study a modern version of the formalism, with an approach entirely based on the form (3) of the interval. Vector spaces in which the length of a vector is given by expressions of the form (3) are called **pseudo-Euclidian spaces**, in opposition with the Euclidian spaces in which the distance is defined by Eq. (2).

3.3 Contravariant and covariant vectors

In Minkowski's geometry, an event is identified by a world point in a four-dimensional continuum. The Cartesian coordinates of the four-dimensional space are labeled from 0 to 3. The basic idea in the treatment of the space-time, which brings a great simplification of the formalism, is to introduce **two types of coordinates for an event, with one set labeled by superscripts, and with the second labeled by subscripts**. Accordingly, in the space-time we can speak about two different representations of vectors [4].

In the three-dimensional Euclidian space we have three-vectors like, for example, the position vector $\vec{r} = (x, y, z)$. By analogy, we shall define the position four-vector in the four-dimensional space in terms of the coordinates, which we denote by

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z.$$

In the above equations the superscripts do not represent powers!

Then the position vector in the space-time is

$$\vec{X} \equiv \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) \equiv X^i.$$

The label (index) i takes the values $i = 0,...,3$.

A vector with superscript labels is called a contravariant vector, and its components are called contravariant components. Hence $X^2 = x^2 (= y)$ is the third contravariant component of the contravariant position vector.

A second representation of the four-dimensional position vector can be given in terms of its covariant components, defined as

$$x_0 = ct \ , \ x_1 = -x^1 = -x, \ x_2 = -x^2 = -y, \ x_3 = -x^3 = -z .$$

The covariant components of the position vector are then

$$\vec{X} \equiv (x_0, x_1, x_2, x_3) \equiv X_i .$$

Therefore $X_1 = x_1 = -x^1 = -x$ is the second covariant component of the position vector.

Under the transformation of inertial frames S->S', the coordinates transform as

$$(X^0)' = \gamma\left(X^0 - \frac{\mathrm{v}}{c}X^1\right), \ (X^1)' = \gamma\left(X^1 - \frac{\mathrm{v}}{c}X^0\right), \ (X^2)' = X^2, \ (X^3)' = X^3,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} .$$

A set of four-numbers (A^0, A^1, A^2, A^3) which have the same transformation rules under a Lorentz transformation as the coordinates is called a contravariant four-vector.

Thus any contravariant four-vector A^i transforms as

$$(A^0)' = \gamma\left(A^0 - \frac{\mathrm{v}}{c}A^1\right), \ (A^1)' = \gamma\left(A^1 - \frac{\mathrm{v}}{c}A^0\right), \ (A^2)' = A^2, \ (A^3)' = A^3.$$

Four vectors can also be generally written as $\vec{A} \equiv A^i = (A^0, \vec{a})$ or $\vec{A} \equiv A_i = (A_0, -\vec{a})$, where $\vec{a}$ is a tri-dimensional vector, called the spatial component of the four-vector. A^0 is called the temporal component.

The scalar product of two four-vectors is defined as

$$\vec{A}\cdot\vec{B} \equiv \sum_{i=0}^{3} A_i B^i = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3 =$$
$$A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 =$$
$$A^0 B^0 - \vec{a}\cdot\vec{b}.$$

At this moment we shall introduce a very important convention, the *Einstein summation convention*, which states that

an index that occurs twice in an expression is summed over.

This convention allows us to eliminate the $\sum$ symbol from a sum, also giving the opportunity of writing expressions containing complicate summations in a very compact form. Hence using the Einstein convention the scalar product of two vectors can be written in a very elegant and simple form as

$$\vec{A}\cdot\vec{B} \equiv A_i B^i .$$

This is identical to $\vec{A}\cdot\vec{B} \equiv \sum_{\nu=0}^{3} A_\nu B^\nu$. The product $A_i B^i$ is a four-scalar, a quantity which is invariant under Lorentz transformations of the four-dimensional coordinate system. The scalar product of the four-dimensional position vector $\vec{X}$ with itself is given by

$$\vec{X}\cdot\vec{X} = X_i X^i = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 =$$
$$c^2 t^2 + (-x)x + (-y)y + (-z)z =$$
$$c^2 t^2 - x^2 - y^2 - z^2 .$$

The expression for the magnitude of the square of the position vector is the same as the interval, given by Eq. (3):

$$s^2 = X_j X^j .$$

In Euclidian three-space, the squared interval between points, or the (differential) distance is

$$ds^2 = dx^2 + dy^2 + dz^2 ,$$

and is invariant under rotations and translations of the rectangular Cartesian coordinate axes.

In four dimensions, an event is a point in space-time. By analogy with three-space, the interval between events in four-space is the "distance" ds^2 defined as

$$ds^2 = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_1 dx^1 + dx_2 dx^2 + dx_3 dx^3,$$

or

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{4}$$

ds^2 is invariant under three-dimensional rotations and translations of the rectangular coordinate axes and with respect to the Lorentz transformation in the plane (ct, x). The property of Lorentz invariance is also true for all scalar products of four-vectors in the four-dimensional space-time.

An arbitrary four-vector $\vec{A} = (A^0, \vec{a})$ can be represented by a displacement vector in space-time. Hence we can classify all four-vectors as

-time-like: $A^2 = A_k A^k = (A^0)^2 - \vec{a}^2 > 0,$

-null: $A^2 = A_k A^k = (A^0)^2 - \vec{a}^2 = 0,$

-space-like: $A^2 = A_k A^k = (A^0)^2 - \vec{a}^2 < 0.$

If the vector $\vec{A}$ represents the four-space trajectory of a free particle, then $\vec{A}$ could be only time-like. If $\vec{A}$ were null, then it could only corresponds to a pulse of light or the trajectory of a massless particle, such as the photon or the neutrino. However, if $\vec{A}$ were to be space-like, then it could not correspond to any particle trajectory.

3.4The four-dimensional velocity and acceleration

From the ordinary three-dimensional velocity vector one can form a four-vector. The four-velocity of a particle is the vector

$$u^i = \frac{dx^i}{ds}.$$

To find its components we note first that according to Eq. (3)

$$ds = cdt\sqrt{1 - \frac{\mathrm{v}^2}{c^2}},$$

where v is the ordinary three-dimensional velocity of the particle.

Thus

$$u^0 = \frac{cdt}{cdt\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}},\ u^1 = \frac{dx^1}{cdt\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \frac{\mathrm{v}_x}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}},\ \text{etc.}$$

Therefore

$$u^i = \left(\frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}, \frac{\vec{\mathrm{v}}}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} \right).$$

Note that the four-velocity is a dimensionless quantity. The contravariant and covariant components of the four-velocity are not independent. It is easy to show that

$$u^i u_i = 1.$$

Similarly to the definition of the four-velocity, the quantity

$$w^i = \frac{d^2 x^i}{ds^2}$$

is called four-acceleration. The four-velocity and four-acceleration vectors are perpendicular:

$$u_i w^i = 0.$$

Exercise. Prove the relations $u^i u_i = 1$ and $u_i w^i = 0$ satisfied by the four-velocity and accelerations vectors.

Exercise. By using the Lorentz transformation properties of the four-velocity find the special relativistic law of the composition of the velocities.

3.5 Four-tensors

A four-dimensional tensor (four-tensor) of the second rank (order) is a set of sixteen quantities, A^{ik}, which under coordinate transformation transform like the products of components of two four-vectors. We similarly define four-tensors of higher rank (order)[6].

The components of a second rank tensor can be written in three forms:

-covariant A_{ik},

-contravariant A^{ik} and

-mixed A_k^i.

The connection between the different types of components is determined from the general rule: raising or lowering a space index (1, 2, 3) changes the sign of the component, while raising or lowering the time index (0) does not.

Thus:

$$A_{00} = A^{00},\ A_{01} = -A^{01},\ A_{11} = A^{11},$$

$$A_0^0 = A^{00},\ A_0^1 = A^{01},\ A_1^0 = -A^{01},\ A_1^1 = -A^{11} \text{ etc.}$$

Under purely spatial transformations the nine quantities A^{11}, A^{12}, … form a three-tensor. The three components A^{01}, A^{02}, A^{03} and the three components A^{10}, A^{20}, A^{30} constitutes three-dimensional vectors, while the component A^{00} is a three-dimensional scalar.

A tensor A^{lm} is said to be symmetric if

$$A^{lm} = A^{ml},$$

and antisymmetric if

$$A^{lm} = -A^{ml}.$$

In an anti-symmetric tensor all the diagonal components (that is the components $A^{00}, A^{11},\ldots$ are zero, since, for example, we must have $A^{00} = -A^{00}$.

Example. Find the law of transformation of the components of a symmetric four-tensor A^{ik} under Lorentz transformations.

A second-order tensor transforms like the product of two four-vectors,

$$A^{Ik} \sim B^i C^k,$$

$$A'^{lk} \sim B'^{i}\, C'^{k}\,,$$

where

$$B'^{0} = \gamma\left(B^{0} - \frac{\mathrm{v}}{c}B^{1}\right),\; B'^{1} = \gamma\left(B^{1} - \frac{\mathrm{v}}{c}B^{0}\right),$$

$$C'^{0} = \gamma\left(C^{0} - \frac{\mathrm{v}}{c}C^{1}\right),\; C'^{1} = \gamma\left(C^{1} - \frac{\mathrm{v}}{c}C^{0}\right).$$

Consequently, for a symmetric second order tensor we obtain the following transformation laws of the components

$$A'^{00} \sim B'^{0}\, C'^{0}\,,$$

$$A'^{00} \sim \gamma^{2}\left(B^{0} - \frac{\mathrm{v}}{c}B^{1}\right)\left(C^{0} - \frac{\mathrm{v}}{c}C^{1}\right)$$

$$A'^{00} \sim \gamma^{2}\left(B^{0}C^{0} - \frac{\mathrm{v}}{c}B^{0}C^{1} - \frac{\mathrm{v}}{c}B^{1}C^{0} + \frac{\mathrm{v}^{2}}{c^{2}}B^{1}C^{1}\right),$$

$$A'^{00} \sim \gamma^{2}\left(\underbrace{B^{0}C^{0}}_{A^{00}} - \frac{\mathrm{v}}{c}\underbrace{B^{0}C^{1}}_{A^{01}} - \frac{\mathrm{v}}{c}\underbrace{B^{1}C^{0}}_{A^{10}} + \frac{\mathrm{v}^{2}}{c^{2}}\underbrace{B^{1}C^{1}}_{A^{11}}\right),$$

$$A'^{00} = \gamma^{2}\left(A^{00} - 2\frac{\mathrm{v}}{c}A^{01} + \frac{\mathrm{v}^{2}}{c^{2}}A^{11}\right).$$

Similarly for the other components we obtain

$$A'^{01} = \gamma^{2}\left[\left(1 + \frac{\mathrm{v}^{2}}{c^{2}}\right)A^{01} - \frac{\mathrm{v}}{c}A^{00} - \frac{\mathrm{v}}{c}A^{11}\right],\; A'^{02} = \gamma\left[A^{02} - \frac{\mathrm{v}}{c}A^{12}\right],$$

$$A'^{11} = \gamma^{2}\left(A^{11} - 2\frac{\mathrm{v}}{c}A^{01} + \frac{\mathrm{v}^{2}}{c^{2}}A^{00}\right),\;\; A'^{12} = \gamma\left(A^{12} - \frac{\mathrm{v}}{c}A^{02}\right),$$

$$A'^{22} = A^{22},\;\; A'^{33} = A^{33} \text{ etc.}$$

Exercise. Find the law of transformation of the components of an anti-symmetric four-tensor A^{ik} under Lorentz transformations.

In every tensor equation, the two sides must contain identical and identically placed (i.e. above or below) free indices (as distinguished from dummy (summation) indices). The free indices in tensor equations can be shifted up or down, but this must be done simultaneously in all terms in the equation. **Equating covariant and contravariant components of different tensors is not allowed, such an equation, even if it happened by chance to be valid in a particular reference system, would be violated on going on another frame.**

From the tensor components A^{ik} one can form a scalar by taking the sum

$$A_i^i = A_0^0 + A_1^1 + A_2^2 + A_3^3.$$

This sum is called **the trace of the tensor** and the operation for obtaining it is called **contraction**.

The formation of the scalar product of two vectors is a contraction operation: it is the formation of the scalar $A^l B_l$ from the tensor $A^l B_k$. In general, contracting on any pair of indices reduces the rank of the tensor by 2. For example, A^i_{kli} is a tensor of second rank, $A^i_k B^k$ is a four-vector, A^{ik}_{ik} is a scalar etc.

The unit four-tensor δ^i_k satisfies the condition that for any four-vector A^i

$$\delta_i^k A^i = A^k.$$

The components of this tensor are

$$\delta_i^k = \begin{cases} 1, \text{if } i = k \\ 0, \text{if } i \neq k \end{cases}.$$

Its trace is $\delta_k^k = 4$.

By raising one index or lowering the other in δ^i_k we can obtain the contra or covariant tensor g^{ik} or g_{ik}, which is called the metric tensor. The tensors g^{ik} and g_{ik} have identical components, which can be written as a matrix:

$$g^{ik} = g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The index i labels the rows and k the columns in the order $0,1,2,3$. It is clear that

$$g_{ik} A^k = A_i, \; g^{ik} A_k = A^i .$$

The tensors δ_k^i, g_{ik}, g^{ik} are special in the sense that their components are the same in all coordinate systems. The completely anti-symmetric tensor of fourth rank e^{iklm} has the same property. **This is the tensor whose components change sign under interchange of any pair of indices and whose non-zero components are** ± 1**.** From the anti-symmetry it follows that all components in which two indices are the same are zero, so that the only non-vanishing components are those for which all four indices are different. We set

$$e^{0123} = +1,$$

and hence $e_{0123} = -1$. Then all the other non-vanishing components e^{iklm} are equal to +1 or -1, according as the numbers i, k, l, m can be brought to the arrangement 0,1,2,3 by an even or an odd number of transpositions. Hence the formal definition of the totally anti-symmetric four-tensor is

$$e_{iklm} = \begin{cases} -1, \text{ for } (iklm) \text{ an even permutation of } (0\,1\,2\,3) \\ 1, \text{ for } (iklm) \text{ an odd permutation of } (0\,1\,2\,3) \\ 0 \text{ otherwise} \end{cases}$$

Some components are explicitly shown below:

$$e_{0123} = e_{2130} = e_{1203} = -e_{3012} = -1,$$

$$e_{1120} = e_{1112} = e_{0230} = 0 .$$

The number of such components is 4!=24. Thus

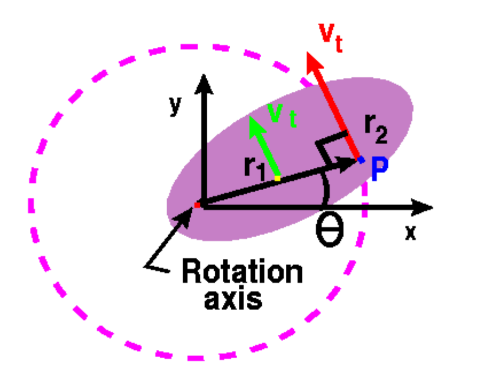
$$e^{iklm} e_{iklm} = -24.$$

With respect to rotations of the coordinate system, the quantities e^{iklm} behave like the components of a tensor; but if we change the sign of one or three of the coordinates the components e^{iklm}, being defined as the same in all coordinate systems, do not change, whereas some of the components of a tensor should change sign. Thus e^{iklm} is, strictly speaking, not a tensor, but rather a pseudotensor.

Pseudotensors of any rank, in particular pseudoscalars, behave like tensors under all coordinate transformations except those that cannot be reduced to rotations, i.e. reflections, which are changes in sign of the coordinates that are not reducible to a rotation.

3. 6 Applications in classical mechanics

We shall discuss now the most important quantities that characterize the motion of a **rigid body**.



Rigid bodies generally are not just **translating**, but **rotating** as well. The point P on the body executes a circular path with instantaneous velocity v_t (t is for transverse). Since the radius of the path, r, is fixed for a rigid body, there is no corresponding v_r or radial velocity (or rather it is always zero). The instantaneous radial **acceleration,** however is not zero. This is allowed because the tangential velocity is constantly changing direction, so there must be some acceleration involved.

The **kinetic energy** T of a rigid body with fix point O is given by the formula

$$T = \frac{1}{2}\int_M \vec{\mathrm{v}} \cdot \vec{\mathrm{v}}\, dm,$$

where the integration is extended over the mass of the body M. The **velocity** can be represented as

$$\vec{\mathrm{v}} = \vec{\omega} \times \vec{r},$$

where $\vec{r}$ is the **radius vector** of the mass element dm with respect to the origin O. Therefore for the kinetic energy we obtain

$$2T = \int_M (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) dm = \int_M \vec{\omega} \cdot (\vec{r} \times \vec{\omega} \times \vec{r}) dm = \int_M \left[|\vec{\omega}|^2 |\vec{r}|^2 - (\vec{\omega} \cdot \vec{r})^2 \right] dm.$$

Let's assume now that the position of the rigid body is described in some curvilinear coordinates. Then

$$2T = \int_M \left[\omega_i \omega^i |\vec{r}|^2 - (\omega_i r^i)(\omega^k r_k)\right] dm = \omega^i \omega_k \int_M \left(\delta_i^k |\vec{r}|^2 - r_i r^k\right) dm = J_i^k \omega^i \omega_k .$$

J_i^k is a tensor, called **the inertia tensor**, and it describes how the mass in a body is distributed with respect to the center of mass.

The **angular momentum** $\vec{l}_O$ of the rigid body with respect to a fixed point O is defined as

$$\vec{l}_O = \int_M \vec{r} \times \vec{\mathrm{v}} dm = \int_M \vec{r} \times (\vec{\omega} \times \vec{r}) dm ,$$

or

$$\vec{l}_O = \int_M \left[\vec{\omega} |\vec{r}|^2 - \vec{r}(\vec{\omega} \cdot \vec{r})\right] dm .$$

In an arbitrary system of coordinates

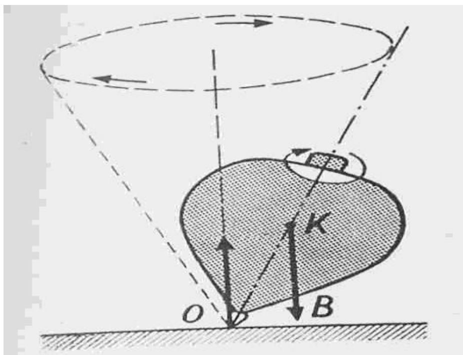
$$l_O^j = \omega^i \int_M \left(\delta_i^j |\vec{r}|^2 - r_i r^j\right) dm = J_i^j \omega^i .$$

Based on the **theorem of the angular momentum**, the **equation of motion** of a rigid body can be given in the form

$$\frac{dl_O^j}{dt} = L_O^j ,$$

where L_O^j are the components of the momentum of the exterior forces applied to the body, with respect to the fixed point O.

These equations are called **the Euler equations**, and are the basic equations in the study of rigid bodies. They are the main mathematical tool for the study of important systems like the gyroscope.



3.7 The four-gradient and the Hodge star operation

The four gradient of a scalar ϕ is the four-vector

$$\frac{\partial \phi}{\partial x^i} = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \nabla \phi \right).$$

The derivatives are the covariant components of the four-vector. The differential of a scalar,

$$d\phi = \frac{\partial \phi}{\partial x^i} dx^i$$

is also a scalar; from its form (scalar product of two four-vectors) it follows that $\frac{\partial \phi}{\partial x^i}$ are the components of a covariant vector. The divergence of a four-vector, $\frac{\partial A^i}{\partial x^i}$, in which we differentiate the contravariant components A^i, is a scalar.

The totally anti-symmetric four-tensor e_{iklm} allows the introduction of an important mathematical operation, called the Hodge * (star) or duality transformation. The * operation transforms tensors of rank p into tensors of rank $(n-p)$, where n is the dimension of the space according to the general rule

$$^{*}F_{i_1 i_2 \ldots i_p} = \frac{1}{(n-p)!} e_{i_1 i_2 \ldots i_p i_{p+1} \ldots i_n} F^{i_{p+1} i_{p+2} \ldots i_n}.$$

The Hodge dual of a second rank tensor $F^{\mu\nu}$ is given, in the four-dimensional space-time, by

$$^{*}F_{\alpha\beta} = \frac{1}{2} e_{\alpha\beta\mu\nu} F^{\mu\nu}.$$

In the same four-dimensional geometry, the Hodge dual of a third order tensor $F^{\mu\nu\sigma}$ is

$$^{*}F_{\alpha} = e_{\alpha\beta\mu\nu} F^{\beta\mu\nu}.$$

3.8 Integration in the four-dimensional space-time

In the three-dimensional space one can extend integrals over a volume, a surface or a curve. In the four-dimensional space there are four types of integration:

a) **Integral over a curve in four-space**. The element of integration is the line element, i.e. the four-vector dx^i.

b) **Integral over a two-dimensional surface in four-space**. In three space the projection of the area of the parallelogram formed from the vectors $d\vec{r}$ and $d\vec{r}\,'$(what is dr' ?)on the coordinate planes $x_\alpha x_\beta$ are $df_{\alpha\beta} = dx_\alpha dx_\beta^{'} - dx_\beta dx_\alpha^{'}$. Analogously, in four-space the infinitesimal element of surface is given by the anti-symmetric tensor of second rank

$$df^{ik} = dx^i dx'^k - dx^k dx'^i .$$

In the three-dimensional space one uses as surface element in place of the tensor $df_{\alpha\beta}$ the vector df_α dual to the tensor $df_{\alpha\beta}$:

$$df_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} df_{\beta\gamma} .$$

Exercise. Show that the components of the vector df_α are the components of the vector product of $d\vec{r}$ and $d\vec{r}\,'$, $d\vec{f} = d\vec{r} \times d\vec{r}\,'$.(?)

Generally in three dimensions df_α is a vector normal to the surface element and equal in absolute magnitude to the area of the element. In four-space we cannot construct such a vector, but we can construct the tensor df^{*ik} dual to the tensor df^{ik},

$$df^{*ik} = \frac{1}{2} e^{iklm} df_{lm} .$$

Geometrically it describes an element of surface equal to and normal to the element of surface df^{ik}.

c) **Integral over a hypersurface, i.e. over a three-dimensional geometric object**. In the three dimensional space the volume of the parallelepiped spanned by three vectors is equal to determinant of the third rank formed from the components of the vectors. One obtains analogously the projections of the volume of the parallelepiped (i.e. the areas of the hypersurface) spanned by three four-vectors dx^i, dx'^i, dx''^i; they are given by the determinant

$$dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix},$$

which forms a tensor of rank 3, anti-symmetric in all three indices. As element of integration over the hypersurface it is more convenient to use the four-vector dS^i, dual to the tensor dS^{ikl}:

$$dS^i = -\frac{1}{6} e^{iklm} dS_{klm}, \; dS_{klm} = e_{iklm} dS^i .$$

Here

$$dS^0 = dS^{123}, \; dS^1 = dS^{023}, \ldots$$

Geometrically, dS^i is a four-vector equal in magnitude to the areas of the hypersurface element, and normal to this element (i.e. perpendicular to all lines drawn in the hypersurface element). In particular, $dS^0 = dxdydz$, i.e. it is the element of the three-dimensional volume dV, the projection of the hypersurface element on the hyperplane $x^0 =$ constant.

Exercise. Show that the components of the four-vector dS^i are given by $\left(dx^1 dx^2 dx^3, dx^0 dx^2 dx^3, dx^0 dx^1 dx^3, dx^0 dx^1 dx^2\right)$.

d) **Integral over a four-dimensional volume**; the element of integration is the scalar

$$d\Omega = dx^0 dx^1 dx^2 dx^3 = cdtdxdydz = cdtdV .$$

The element is a scalar; it is obvious that the volume of a portion of four-space is unchanged by a rotation of the coordinate system.

Analogous to the theorems of Gauss and Stokes in three-dimensional vector analysis, there are theorems that enable us to transform four-dimensional integrals.

The integral over a closed hypersurface can be transformed into an integral over the four-volume contained within it by replacing the element of integration dS_i by the operator

$$dS_i \to d\Omega \frac{\partial}{\partial x^i} .$$

For example, for the integral of a vector A^i we have

$$\oint A^i dS_i = \int \frac{\partial A^i}{\partial x^i} d\Omega .$$

This formula is the generalization of the Gauss theorem. An integral over a two-dimensional surface is transformed into an integral over the hypersurface spanning it by replacing the element of integration df^*_{ik} by the operator

$$df^*_{ik} \to dS_i \frac{\partial}{\partial x^k} - dS_k \frac{\partial}{\partial x^i}.$$

For example, for the integral of the anti-symmetric tensor A^{ik} we have

$$\frac{1}{2}\int A^{ik} df^*_{ik} = \frac{1}{2}\int\left(dS_i \frac{\partial A^{ik}}{\partial x^k} - dS_k \frac{\partial A^{ik}}{\partial x^i}\right) = \int dS_i \frac{\partial A^{ik}}{\partial x^k}.$$

The integral over a four-dimensional closed curve is transformed into an integral over the surface spanning it by the substitution:

$$dx^i \to df^{ki} \frac{\partial}{\partial x^k}.$$

Thus for the integral of a vector we have

$$\oint A_i dx^i = \int df^{ki} \frac{\partial A_i}{\partial x^k} = \frac{1}{2}\int df^{ik}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right).$$

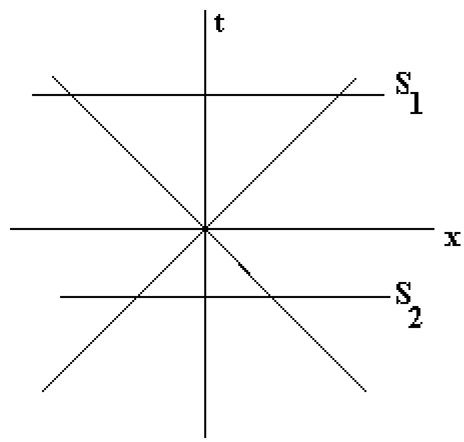
Finally we shall prove the following important

Theorem. If the four-vector A^i is divergence-free, $\frac{\partial A^i}{\partial x^i} = 0$, then $\int_S A^i dS_i$ is independent of the space-like surface S at all times.

From the Gauss theorem we obtain first

$$\oint A^i dS_i = \int \frac{\partial A^i}{\partial x^i} d\Omega = 0.$$

If we now take S to consist of any two space-like surfaces S_1 and S_2, and if we assume A^i approaches zero sufficiently rapidly, we can neglect the contribution to S of the infinite cylinder connecting S_1 and S_2.Thus we obtain



$$\oint A^i dS_i = \int_{S_2} A^i dS_i - \int_{S_1} A^i dS_i = 0.$$

Hence $\int_S A^i dS_i$ is independent of the space-like surface S at all times. For example, by choosing the hypersurface $x^0 = 0$ we obtain

$$\oint A^i dS_i = \int_S A^0 dS_0 = \int_V A^0 dV = const.$$

3.9 The geometrical meaning of the metric tensor

Previously we have shown that the squared interval between two events in a Minkowskian space-time can be written as

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

This expression is a special case of a more general result used in differential geometry for a path length[7]. **The general form for the length of a path at a point P in a vector space of n dimensions is**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad (6)$$

where both repeated indices μ and ν are summed from zero up to $n-1$. **Vector spaces in which such a representation is possible are called Riemannian spaces.**

The expression (6) for ds^2 is known as a metric and $g_{\mu\nu}$ is called the metric tensor[8]. **The metric tensor has the following general properties**:

1. $g_{\mu\nu}$ is a n-row $\times$ n-column array of coefficients.

2. The values of the coefficients may depend on the position of the point P.

3. The nature of the metric tensor may depend on the nature of the vector space and the choice of the coordinate system.

4. The metric tensor is symmetric, $g_{\mu\nu} = g_{\nu\mu}$.

Let's try to obtain the form of the metric tensor appropriate to the Euclidian space. In such a space

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2. \tag{7}$$

It is then easy to check that Eq. (6) reduces to Eq. (7) if and only if first $g_{\mu\nu} = 0$ for cases where $\mu \neq \nu$ (in other words the array of coefficients has to form a diagonal matrix) and second, if

$$g_{11} = g_{22} = g_{33} = 1.$$

These two conditions are in fact just the definition of either the Kronecker delta symbol $\delta_{\mu\nu}$ or of the unit matrix. That is, we may write the metric tensor for the Euclidian three-space as either

$$g_{\mu\nu} = \delta_{\mu\nu},$$

or as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In a more compact notation we can also write $g_{\mu\nu} = \mathrm{diag}(1,1,1)$.

Now let's turn to the Minkowski space. In this space the metric tensor has a special notation, being usually denoted by $\eta_{\mu\nu}$. Again, $\eta_{\mu\nu}$ must be diagonal and the diagonal elements must satisfy the conditions

$$\eta_{00} = 1,\ \eta_{11} = -1,\ \eta_{22} = -1,\ \eta_{33} = -1,$$

or

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1).$$

The metric tensor acts as a raising or lowering operator on the vector indices in the four-space, transforming a contravariant component to a covariant one and vice-versa, by means of the relation

$$X_\mu = \eta_{\mu\nu} X^\nu .$$

Thus

$$X_0 = \eta_{0\nu} X^\nu, \;\; X_0 = \eta_{00} X^0, \;\; X_0 = X^0,$$

$$X_1 = \eta_{1\nu} X^\nu, \;\; X_1 = \eta_{11} X^1, \;\; X_1 = -X^1,$$

etc.

Notes

[1] The mathematical theory of the vector spaces and differential geometry are indispensable tools for the study of modern physical theories like quantum mechanics and relativity. A good introduction to these fields of mathematics and some of their physical applications can be found in S. Hassani, Mathematical physics: a modern introduction to its foundations, New York, Springer, 1999.

[2] For a discussion of vector spaces see also W. D. McComb, Dynamics and relativity, Oxford, Oxford University Press, 1999.

[3] At an elementary level the properties of the space-time continuum are presented in E. F. Taylor and J. A. Wheeler, Space-time physics, W. H. Freeman and Company, New York, 2001, where the physical meaning of the relativistic interval and of the light cone with many applications is discussed in great detail.

[4] The presentation of the properties of the covariant and contravariant vectors follows W. D. McComb, Dynamics and relativity, Oxford University Press, Oxford, 1999 and L. D. Landau and E. M. Lifshitz, The classical theory of fields, Oxford, Butterworth-Heinemann, 1998.

[5] The definitions of the proper line element, four-velocity and four-acceleration follow the treatment in L. D. Landau and E. M. Lifshitz, The classical theory of fields, Pergamon Press, Oxford, 1975; see also W. D. McComb, Dynamics and relativity, Oxford, Oxford University Press, 1999.

[6] The presentation of the properties of the special relativistic four-vectors and tensors is based on L. D. Landau and E. M. Lifshitz, The classical theory of fields, Pergamon Press, Oxford, 1975

[7] The metric tensor and its properties are discussed in great detail, from a physical point of view, in L. D. Landau and E. M. Lifshitz, The classical theory of fields, Pergamon Press, Oxford, 1975 and W. D. McComb, Dynamics and relativity, Oxford University Press, Oxford, 1999. For a more advanced mathematical approach see S. Hassani, Mathematical physics: a modern introduction to its foundations, Springer, New York, 1999.