PHYS 3033 GENERAL RELATIVITY PART I
Chapter 4
Special relativistic dynamics

4.1 Elementary Lagrangian and Hamiltonian Dynamics

Let's consider the Newton equation

$$\vec{F} = m\vec{a}$$

in the one-dimensional case, in which all the physical parameters depend on the variable x only. Assume that the force is conservative, which means that it is given as the spatial derivative of the potential energy $U(x)$:

$$F = -\frac{dU}{dx}.$$

Thus for this case Newton's equation can be re-written as

$$m\ddot{x} = -\frac{dU}{dx}, \tag{1}$$

where

$$\dot{x} = \frac{dx}{dt} = \mathrm{v}$$

and

$$\ddot{x} = \frac{d^2x}{dt^2} = a.$$

Define the **Lagrangian** $L(x,\dot{x})$ as a function of two variables, the position x and the speed $\dot{x}$,

$$L(x,\dot{x}) = T(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - U(x), \tag{2}$$

where the kinetic energy $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$ is a function of the speed variable only and the potential energy is a function of the position only.

From Eq. (2) one can easily see that

$$\frac{\partial L}{\partial x} = -\frac{dU}{dx}, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p. \tag{3}$$

Then obviously

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}. \qquad (4)$$

With the use of Eq. (3) Newton's Eq. (1) becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

which is called the **Euler-Lagrange equation in one dimension** [1].

To find the **Hamiltonian formulation of dynamics**, we define first the Hamiltonian $H(x, p)$ as a function of two new variables, the momentum p and the position x:

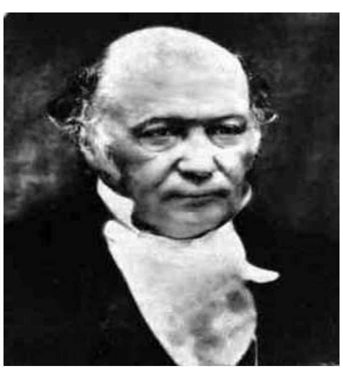
$$H(x, p) = p\dot{x} - L(x, \dot{x}),$$

which is just the total energy $T + U$ as

$$H(x, p) = p\dot{x} - L(x, \dot{x}) = m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - U(x)\right) = T + U. \qquad (5)$$

The Hamilton equations, which replace Newton's equation of motion Eq. (1) are given by

$$\dot{x} = \frac{\partial H}{\partial p}, \; \dot{p} = -\frac{\partial H}{\partial x}. \qquad (6)$$



Sir William Rowan Hamilton (1805-1865)

Hamilton is one of the most important scientists of the 19th century. As a child, he learned 14 languages and taught himself mathematics at age 17. Hamilton developed the mathematical theory of quaternions, which is an anti-commutative algebra, with important applications to quantum mechanics. Perhaps Hamilton's most important contribution came from his reformulation of Newton's Laws. In the same way that Lagrange provided a new method for solving mechanical problems, Hamilton developed an alternative formalism. He showed that the results were equivalent in the three methods, but his proves to be most useful for a certain class of problems. Even today, Hamiltonian Mechanics is used to determine orbital trajectories of satellites.

4.2 The principle of least action

In 1746 the French scientists Maupertuis formulated the **Principle of Least Action**, which is generally credited to one of the three great scientists, Euler, Lagrange, and Hamilton, who further developed it. This Principle is one of the greatest generalizations in all physical science, although not fully appreciated until the advent of quantum mechanics in the last century. Maupertuis arrived at this principle from a feeling that the very perfection of the universe demands a certain economy in nature and is opposed to any needless expenditure of energy. **Natural motions must be such as to make some quantity a minimum.**


Maupertuis (1698-1759)

In 1732 Maupertuis introduced Newton's theory of gravitation to France. He was a member of an expedition to Lapland in 1736, which set out to measure the length of a degree along the meridian. His measurements verified Newton's predictions that the Earth would be an oblate spheroid. Maupertuis published on many topics, including mathematics, geography, astronomy and cosmology. In 1744 he first enunciated the Principle of Least Action and he published it in Essai de cosmologie (Cosmological Essays) in 1750. He hoped that the principle might unify the laws of the universe.

The formal definition of the principle of least action is that it is **"the principle stating that the actual motion of a conservative dynamical system between two points takes place in such a way that a function of the coordinates and velocities, called the action, has a minimum value with reference to all other paths between the points which correspond to the same energy."** [2]

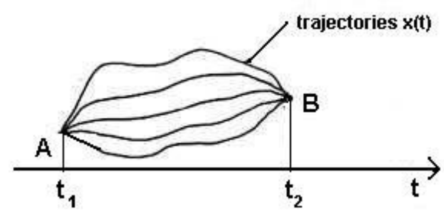
The action S of a system, which is a scalar quantity, is defined as the time integral of the Lagrangian function,

$$S = \int_{t_1}^{t_2} L\left(x(t), \dot{x}(t)\right) dt.$$

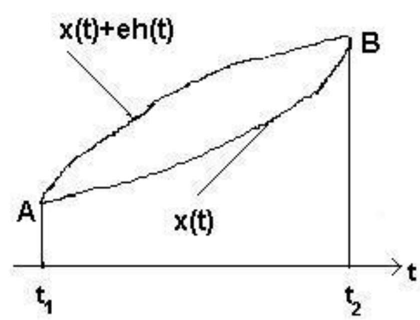
Therefore the problem of the motion of a mechanical system can be stated as that of finding the path $x(t), t_1 \le t \le t_2$, such that the action S is minimal. **From a mathematical point of view this class of problems belongs to the field of mathematics called variational calculus and a quantity defined like the action is named functional** [2] . **A functional will have its extremal value when its**

"variation" is equal to zero. (This is like a function having an extremal value when its derivative is equal to zero).

Consider two points A and B. **There are many trajectories joining the points, but a mechanical system which is evolving between them is choosing the trajectory that makes the action functional extremal**.



Let's assume that $x(t)$ is the real trajectory of the body. Let's us also imagine a second trajectory, which is very near to the first, given by $x(t)+\varepsilon h(t)$, where $h(t)$ is an arbitrary



time dependent function and ε is a constant satisfying the condition $\varepsilon << 1$. Obviously at the ends of all varied paths $h(t)$ satisfies the conditions

$$h(t_1) = h(t_2) = 0. \tag{7}$$

The value of the action integral, which is a scalar, will be necessarily different according to the path taken by the particle to go from A to B,

$$S[x(t)] \neq S[x + \varepsilon h(t)].$$

To find the curve or curves and the path or paths that make the action extremal, we shall use the variation of the trajectory and evaluate the action for the extremal and for the varied trajectory. For the Lagrangian along the varied path we obtain, by using a Taylor series expansion,

$$L\left(x+\varepsilon h, \dot{x}+\varepsilon \dot{h}\right)=L\left(x, \dot{x}\right)+\varepsilon\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right).$$

Therefore the variation of the action along the two paths is given by

$$S\left[x+\varepsilon h(t)\right]-S\left[x(t)\right]=\varepsilon\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right)dt .$$

The second term in the integrand can be transformed by using partial integration,

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{x}}\dot{h}dt=\left[\frac{\partial L}{\partial \dot{x}}h(t)\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)h(t)dt=-\int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)h(t)dt ,$$

since $h\left(t_1\right)=h\left(t_2\right)=0.$

Then the variation of the action integral for a very small ε is

$$\frac{\partial S}{\partial \varepsilon}=\lim_{\varepsilon\to 0}\frac{S\left[x+\varepsilon h(t)\right]-S\left[x(t)\right]}{\varepsilon}=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)h(t)dt.$$

Since $h(t)\neq 0$ for $t_1<t<t_2$, for the path $x(t)$ followed by the particle between the two fixed points A and B be an extremal of the action S, it is necessary and sufficient that the quantity between the brackets in the integral be zero. Then this condition gives the Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=\frac{\partial L}{\partial x}.$$

Therefore the equation of motion of a particle under the action of a conservative force can be derived from the Principle of the Least Action.

Lagrange (1736-1813)

The work of Lagrange, one of the most important scientists of the eighteenth century, covered many topics-astronomy, mathematics, mechanics etc. He applied the calculus of variations to mechanics. In a work on the foundations of dynamics, “Mécanique analytique” (Analytical Mechanics), published in 1788, Lagrange based his development on the principle of least action and on kinetic energy. The “Mécanique analytique” summarized all the work done in the field of mechanics since the time of Newton, and is notable for its use of the theory of differential equations.

4.3 The relativistic Lagrangian and Hamiltonian for a free particle

To determine the action integral for a free relativistic particle (a particle not under the influence of any external force) we note that this integral must not depend on our choice of reference system, that is, it must be invariant under Lorentz transformations[3]. Then it follows that it must depend on a scalar. Furthermore, it is clear that the integrand must be a differential of the first order. But the only scalar of this kind that one can construct for a free particle is the interval ds or αds, where α is a constant. Therefore for a free relativistic particle the action must have the form

$$S = -\alpha \int_a^b ds , \qquad (8)$$

where $\int_a^b$ is an integral along the world line of the particle between two particular events of the arrival of the particle at the initial position and at the final position at definite times t_1 and t_2. The action integral can also be represented as an integral with respect to the time,

$$S = \int_{t_1}^{t_2} L dt , \qquad (9)$$

where L represents the Lagrange function of the system. By comparing Eqs. (8) and (9) it immediately follows

$$L = -\alpha c \sqrt{1 - \frac{\mathrm{v}^2}{c^2}} . \qquad (10)$$

The quantity α characterizes the particle. In classical mechanics each particle is characterized by its mass m_0. Let us find the relation between α and m_0. It can be determined from the fact that in the limit $c \to \infty$ the relativistic expression for L must go over into the classical expression $L = m_0 \mathrm{v}^2 / 2$. By expanding the Lagrangian (10) in powers series of v/c we find

$$L = -\alpha c \sqrt{1 - \frac{\mathrm{v}^2}{c^2}} \approx -\alpha c + \frac{\alpha \mathrm{v}^2}{2c} .$$

Constant terms in the Lagrangian do not affect the equation of motion and can be omitted. By comparing with the classical Lagrangian we obtain $\alpha = m_0 c$. Thus the action and the Lagrangian for a free relativistic material point are

$$S = -m_0 c\int_a^b ds\,,\; L = -m_0 c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,.$$

By the momentum of a particle we mean the vector

$$\vec{p} = \frac{\partial L}{\partial \vec{\mathrm{v}}}\,.$$

By using the Lagrangian for the relativistic particle we find

$$\vec{p} = \frac{m_0\vec{\mathrm{v}}}{\sqrt{1-\frac{\vec{\mathrm{v}}^2}{c^2}}}\,. \tag{11}$$

For small velocities this expression goes over into the classical relation $\vec{p} = m_0\vec{\mathrm{v}}$.
The Hamiltonian of the free particle can be found from

$$H = E = \vec{p}\cdot\vec{\mathrm{v}} - L = \frac{m_0\vec{\mathrm{v}}^2}{\sqrt{1-\frac{\vec{\mathrm{v}}^2}{c^2}}} + m_0c^2\sqrt{1-\frac{\vec{\mathrm{v}}^2}{c^2}} \;\; = \frac{m_0c^2}{\sqrt{1-\frac{\vec{\mathrm{v}}^2}{c^2}}}\,. \tag{12}$$

This very important formula shows that in relativistic mechanics the energy of a free particle does not go to zero for v=0, but rather takes on a finite value $E = m_0c^2$**.**

The Hamiltonian is a function of the momentum and position. Hence we must express it as a function of $\vec{p}$. With the use of Eq. (11) and (12) we obtain

$$H = c\sqrt{\vec{p}^2 + m_0^2c^2}\,.$$

In the classical limit we obtain easily

$$\lim_{c\to\infty} H = m_0c^2 + \frac{p^2}{2m_0},$$

which is the Hamiltonian of a free non-relativistic particle also containing the rest energy contribution.

The same results can be obtained by using the four-dimensional formalism. According to the principle of least action

$$\delta S = -m_0 c\delta \int_a^b ds = -m_0 c\delta \int_a^b \sqrt{dx_i dx^i} =$$

$$-m_0 c\int_a^b \frac{(\delta dx_i)dx^i + dx_i(\delta dx^i)}{2\sqrt{dx_i dx^i}} = -m_0 c\int_a^b \frac{dx_i d\delta x^i}{ds} = -m_0 c\int_a^b u_i d\delta x^i,$$

where we have used the result that the operators d and δ commute. Integrating by parts we obtain

$$\delta S = -m_0 c\int_a^b u_i d\delta x^i = -m_0 c u_i \delta x^i \Big|_a^b + m_0 c\int_a^b \delta x^i \frac{du_i}{ds} ds\,. \qquad (13)$$

Since the variations of δx^i at the two limits corresponding to a and b are zero, $(\delta x^i)_a = (\delta x^i)_b = 0$, we obtain the equation of motion of the relativistic particle in the form

$$\frac{du_i}{ds} = 0\,.$$

To determine the general variation of the action as a function of the coordinates, one must consider the point a as fixed, so that $(\delta x^i)_a = 0$. The second point is to be considered as variable, but only actual trajectories are admissible, i.e. only those which satisfy the equation of motion. Therefore the integral in the Eq. (13) is zero. In place of $(\delta x^i)_b$ we may write simply δx^i. Thus we obtain

$$\delta S = -m_0 c u_i \delta x^i. \qquad (14)$$

The four-vector

$$p_i = -\frac{\partial S}{\partial x^i}$$

is called **the momentum four-vector**. From Eq. (14) we see that the components of the four-momentum of a free particle are

$$p^i = m_0 c u^i = \left(\frac{E}{c}, \vec{p}\right). \quad ?$$

Thus in relativistic mechanics momentum and energy are the components of a single four-vector. The square of the four-momentum satisfies the relation

$$p^i p_i = m_0^2 c^2.$$

4.4 The energy- momentum tensor

Previously we have derived, by using the Lagrangian formalism and the principle of least action, an expression for the energy and momentum of a free particle. Now, having in mind later applications to the case of the gravitational field, we shall extend the derivation to a general form, without taking explicitly into account the nature of the system.

Hence we shall consider any system whose action integral is given in the general form

$$S = \int \Lambda\left(q, \frac{\partial q}{\partial x^i}\right) dV dt = \frac{1}{c}\int \Lambda d\Omega ,$$

where Λ is some function of the quantities q, describing the state of the system and of their first derivatives with respect to the coordinates and time.

The space integral $\int \Lambda dV$ is the Lagrangian of the system, so that Λ is sometimes called Lagrangian density. The mathematical expression of the fact that the system is closed is the absence of any explicit dependence of Λ on the coordinates x^i (this situation is similar to the situation of a closed system in mechanics, where the Lagrangian for a conservative system does not depend explicitly on time).

First we shall derive the equations of motion of the system, by using the principle of least action. They are obtained by varying the action. In the following for brevity we shall denote $q_{,i} \equiv \frac{\partial q}{\partial x^i}$, that is, a comma denotes the partial derivative with respect to the i-th component of the coordinate four-vector.

The variation of the action is then given by

$$\delta S = \frac{1}{c}\int \left(\frac{\partial \Lambda}{\partial q}\delta q + \frac{\partial \Lambda}{\partial q_{,i}}\delta q_{,i} \right) d\Omega = \frac{1}{c}\int \left[\frac{\partial \Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial \Lambda}{\partial q_{,i}}\delta q \right) - \delta q \frac{\partial}{\partial x^i}\frac{\partial \Lambda}{\partial q_{,i}} \right] d\Omega = 0 .$$

The second term in the integrand, after transformation by Gauss' theorem, vanishes upon integration over all space. Therefore the equations of motion of the system (the Lagrange equations) are given by

$$\frac{\partial}{\partial x^i}\frac{\partial \Lambda}{\partial q_{,i}} - \frac{\partial \Lambda}{\partial q} = 0 . \qquad (15)$$

Now let's consider the derivative of the Lagrangian density with respect to the coordinates:

$$\frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q}\frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i} . \qquad (16)$$

Substituting $\frac{\partial \Lambda}{\partial q}$ from the Lagrange equations (15) and noting that $q_{,k,i} = q_{,i,k}$ we can transform (16) to the form

$$\frac{\partial \Lambda}{\partial x^i} = \left(\frac{\partial}{\partial x^k} \frac{\partial \Lambda}{\partial q_{,k}} \right) \frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial q_{,k}} \frac{\partial q_{,k}}{\partial x^i} = \frac{\partial}{\partial x^k} \left(q_{,i} \frac{\partial \Lambda}{\partial q_{,k}} \right). \qquad (17)$$

On the other hand we can write

$$\frac{\partial \Lambda}{\partial x^i} = \delta_i^k \frac{\partial \Lambda}{\partial x^k},$$

so that introducing the notation

$$T_i^k = q_{,i} \frac{\partial \Lambda}{\partial q_{,k}} - \delta_i^k \Lambda,$$

we can express Eq. (17) in the form

$$\frac{\partial T_i^k}{\partial x^k} = 0. \qquad (18)$$

An equation of the form $\partial A^k / \partial x^k = 0$ is equivalent with the statement that the integral $\int A^k dS_k$ of the vector over a hypersurface which contains all of three-dimensional space is conserved. It is clear that an analogous result holds for the divergence of a tensor; therefore Eq. (18) asserts that the vector

$$P^i = const \int T^{ik} dS_k$$

is conserved.

This vector must be identified with the four-vector of momentum of the system. We choose the constant factor in front of the integral so that, in accord with the usual definition of the momentum four-vector, the time component P^0 is equal to the energy of the system multiplied by $1/c$. To do this we note that

$$P^0 = \text{const} \int T^{0k} dS_k = \text{const} \int T^{00} dS_0 = \text{const} \int T^{00} dV = \frac{1}{c} \int T^{00} dV,$$

if the integration is extended over the hyperplane $x^0 = \text{const}$. This quantity must be considered **the total energy of the system** and therefore T^{00} is **the energy density** of the system. Thus finally we obtain for the four-momentum of the system the expression

$$P^i = \frac{1}{c}\int T^{ik} dS_k . \qquad (19)$$

The tensor T^{ik} is called **the energy-momentum tensor of the system**.

If we carry out the integration in Eq. (19) over the hyperplane $x^0 = \text{const}$. we obtain

$$P^i = \frac{1}{c}\int T^{i0} dV ,$$

where the integration extends over the whole three-dimensional space. **The space components of P^i form the three-dimensional momentum vector of the system, and the time component is its energy multiplied by $1/c$.** Thus the vector with components

$$\frac{1}{c}T^{10}, \frac{1}{c}T^{20}, \frac{1}{c}T^{30},$$

may be called **the momentum density**. $W = T^{00}$ **is the energy density**.

To clarify the physical meaning of the remaining components of T^{ik}, we separate the conservation equation (18) into space and time components:

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0\alpha}}{\partial x^\alpha} = 0 , \quad \frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 . \qquad (20)$$

We integrate these equations over a volume V in space. From the first equation we obtain

$$\frac{\partial}{\partial t}\int T^{00} dV = -c\int \frac{\partial T^{0\alpha}}{\partial x^\alpha} dV = -c\oint T^{0\alpha} df_\alpha ,$$

where the second integral has been transformed by means of the Gauss theorem(Converts surface integral n to line integral in 4-D space). The surface integral is taken over the surface surrounding the volume V and (df_x, df_y, df_z) are the components of the three-vector of the surface element $d\vec{f}$. The expression on the left is the rate of change of the energy contained in the volume V ; from this it is clear that the expression on the right **is the amount of energy transferred across the boundary of the volume V,** and the vector $\vec{S}$ with components

$$cT^{01}, cT^{02}, cT^{03},$$

is its flux density-**the amount of energy passing through unit surface in unit time.**

There is a definite connection between the energy flux and the momentum density: **the energy flux density is equal to the momentum density multiplied by c^2.**

From the second equation in (20) we find similarly

$$\frac{1}{c}\frac{\partial}{\partial t}\int T^{\alpha 0}dV = -\int \frac{\partial T^{\alpha\beta}}{\partial x^{\beta}}dV = -\oint T^{\alpha\beta}df_{\beta}.$$

On the left we have the change of the momentum of the system in the volume V per unit time. Therefore $\oint T^{\alpha\beta}df_{\beta}$ **is the momentum emerging from the volume V per unit time.**

Thus the components $T^{\alpha\beta}$ of the energy momentum tensor constitute the three-dimensional tensor of momentum flux density: we denote it by $-\sigma_{\alpha\beta}$ (a component of this tensor is the amount of the α-component of the momentum passing per unit time through unit surface perpendicular to the x^{β} axis). $-\sigma_{\alpha\beta}$ **is also called the stress tensor.**

The meanings of the individual components of the energy-momentum tensor are represented in the following table

$$T^{ik} = \begin{pmatrix} W & \frac{S_x}{c} & \frac{S_y}{c} & \frac{S_z}{c} \\ \frac{S_x}{c} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{S_y}{c} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{S_z}{c} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}.$$

4.5 The energy-momentum tensor for macroscopic bodies

The flux of the momentum through the element $d\vec{f}$ of the surface of a body is just the force acting on the surface element. Therefore $-\sigma_{\alpha\beta}$ is the α-component **of the force acting on the element**. Now we introduce a reference system in which a given element of volume of the body is at rest. In such a reference system Pascal's law is valid, **that is the pressure p applied to a given portion of the body is transmitted equally in all directions and is everywhere perpendicular to the surface on which it acts.**

Therefore we can write

$$\sigma_{\alpha\beta}df_{\beta} = -p\,df_{\alpha},$$

so that

$$\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}.$$

As for the components $T^{\alpha 0}$, which represent the momentum density, they are equal to zero for the given volume element in the reference system we are using. The component T^{00} is always the energy density of the body, which we denote by ε; ε / c^2 is the mass density of the body, i.e. the mass per unit volume.

Thus, in the reference system under consideration (in which the body is at rest) the energy momentum tensor has the form

$$T^{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{21}$$

Now it is easy to find the expression of the energy-momentum tensor in an arbitrary reference system. We introduce the four-velocity u^i for the macroscopic motion of an element of volume of the body. In the rest frame of the particular element $u^i = (1,0)$. The expression of T^{ik} must be chosen so that in this reference system it takes the form (21). It is easy to verify that this is

$$T^{ik} = (\varepsilon + p) u^i u^k - p g^{ik},$$

or, for the mixed components

$$T_i^k = (\varepsilon + p) u_i u^k - p \delta_i^k. \tag{22}$$

For a cloud of particles, the pressure, determined by the energy of the microscopic motion of the particle, is much smaller as compared with the rest energy of the cloud, $p << \varepsilon$. In this case the energy-momentum tensor is given by

$$T_i^k = \varepsilon u_i u^k.$$

Taking the trace of the above equation we obtain

$$T_i^i = \varepsilon u_i u^i = \varepsilon \geq 0. \tag{23}$$

According to this formula we shall assume that for every system

$$T_i^i \geq 0. \tag{24}$$

Taking the trace of Eq. (22) we find

$$T_i^i = (\varepsilon + p) u_i u^i - p \delta_i^i = \varepsilon + p - 4p = \varepsilon - 3p \geq 0.$$

Therefore the general property (24) of the energy-momentum tensor for an arbitrary system shows that the following inequality is always valid for the pressure and density of a macroscopic body:

$$p \le \frac{\varepsilon}{3}.$$

4.6 An elementary introduction of the energy-momentum tensor

In the search for the relativistic theory of gravitation the energy density plays an important role. But it makes little sense to consider energy by itself, because what is energy density in one reference frame will be some combination of energy density, energy flux density and momentum flux density as seen from other reference frame. Hence all these quantities must be considered together.

It will be best to explain this for a system consisting of a collection of non-interacting particles (a cloud of dust)[4]. Suppose that near some point in this cloud the density of particles is n per unit volume and their velocity is $\vec{\mathrm{v}}$. In this case the energy density can be expressed in terms of the density and the velocity of the particles:

$$T^{00} = \frac{n m_0 c^2}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}. \qquad (25)$$

This is simply **the product of the number n of particles per unit volume by the energy per particle.** We use the notation T^{00} for energy density for reasons which will soon be obvious.

The **energy flux density** can be defined in the following way: **the energy flux in the x- direction is the amount of energy transported in unit time across a unit yz-area**, that is

$$\frac{n m_0 c^2 \mathrm{v}_x}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}.$$

This is **the product of particle current $n\mathrm{v}_x$ by the energy per particle**. In general, the energy flux density in the k-direction is

$$T^{k0} = \frac{n m_0 c^2 \mathrm{v}^k}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}. \qquad (26)$$

$T^{k0} = T^{0k}$ can also be regarded as the density of the momentum.

Finally, let us define the **momentum flux**. The xy **momentum flux is defined as the amount of** x**-momentum that flows in the** y**-direction per unit area and unit time**. Since the x-momentum per particle is

$$\frac{m_0 \mathrm{v}_x}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}},$$

the xy-momentum flux must obviously be

$$\frac{m_0 \mathrm{v}_x}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} n\mathrm{v}_y .$$

The general expression for the kl-momentum flux is

$$T^{kl} = \frac{n m_0 \mathrm{v}^k \mathrm{v}^l}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} . \qquad (27)$$

It is now easy to show that the **16 components object** T^{ik} **given by Eqs. (25), (26) and (27) is a tensor under Lorentz transformations**. We can proceed as follows. T^{ik} can also be written as

$$T^{ik} = n_0 m_0 c^2 u^i u^k \ ,$$

where we have introduced the quantity n_0 by means of the transformation

$$n_0 = \frac{n}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} .$$

The quantity n_0 is **the particle density as measured in a reference frame that moves with the particles**. This is a consequence of the well-known volume contraction effect of special relativity: a volume containing a given number of particles and moving with them is contracted as measured in the laboratory frame; hence the laboratory density is increased.

Because n_0, also called **the proper particle density**, is a number measured in the local rest frame, it is an invariant (a scalar). Hence $T^{ik} = n_0 m_0 c^2 u^i u^k$ is the product of the scalar n_0 by the tensor $m_0 c^2 u^i u^k$. Therefore T^{ik} is also a tensor, **the energy-momentum tensor**. Its definition can also be expressed as

$$T^{ik} = \rho_0 c^2 u^i u^k ,$$

where $\rho_0 = n_0 m_0$ is **the mass density** as measured in the local rest frame of the particles; ρ_0 is also called **the proper mass density**. From its definition it is obvious that the energy-momentum tensor is symmetric:

$$T^{ik} = T^{ki} .$$

The definition of T^{ik} can be summarized as follows:

a) T^{00} is **the energy density**

b) cT^{0i} is **the energy flow per unit area parallel to the i direction**. In the case of the dust cloud this would constitute heat flow

c) T^{ii} is **the flow of momentum component i per unit area in the i direction**, i.e. the **pressure** across the i plane

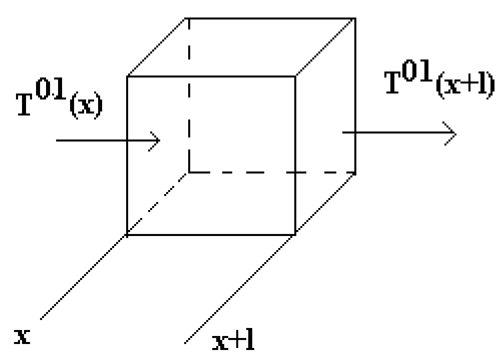
d) T^{ij} is **the flow of the i component of momentum per unit area in the j direction**.

e) cT^{i0} is **the density of the i component of the momentum**.

The energy-momentum tensor embodies a compact description of energy and momentum density.

Conservation laws for energy and momentum take particularly simple forms when expressed in terms of the energy-momentum tensor.

The conservation of the energy offers a good example of the way that this happens. In the figure we can see a cube of edge length l with its edges parallel to the axes Ox, Oy and Oz, respectively, in a medium whose energy-momentum tensor is T^{ik}.



The **rate of change of the energy content** of the box is

$$l^3 \frac{\partial T^{00}}{\partial t}.$$

This change is produced by the net energy inflow through the six faces of the cube. The energy flow through the faces at $x = x$ and $x = x + l$ is, respectively

$l^2 c T^{01}(x)$ inward per unit time

$l^2 c T^{01}(x+l)$ outward per unit time

The net flow inward from these two faces is

$$l^2 c\left[T^{01}(x) - T^{01}(x+l)\right] = -l^3 c \frac{\partial T^{01}}{\partial x}.$$

There are similar contributions from the other pairs of faces:

$$-l^3 c \frac{\partial T^{02}}{\partial y} \text{ and } -l^3 c \frac{\partial T^{03}}{\partial z}.$$

Summing the three contributions gives the total inflow, and so we have

$$l^3 \frac{\partial T^{00}}{\partial t} = -l^3 c\left(\frac{\partial T^{01}}{\partial x} + \frac{\partial T^{02}}{\partial y} + \frac{\partial T^{03}}{\partial z}\right),$$

which can be arranged to become

$$\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{0\alpha}}{\partial x^\alpha} = \frac{\partial T^{0k}}{\partial x^k} = 0. \tag{28}$$

The type of derivative appearing in Eq. (28) is known as the divergence of T^{0k}. A similar procedure applied to the conservation of linear momentum yields the three equations

$$\frac{\partial T^{i\alpha}}{\partial x^\alpha} = 0, i = 1,2,3.$$

Finally all the conservations laws can be combined into a single equation:

$$\frac{\partial T_i^k}{\partial x^k} = 0.$$

This means that **the divergences of the energy-momentum tensor vanish everywhere.**

Notes

[1] Detailed presentations of the Lagrange and Hamilton formalisms in mechanics can be found in standard textbooks, like L.D. Landau and E. M. Lifshitz, Mechanics, Pergamon Press, Oxford, 1976 or H. Goldstein, Classical mechanics, Addison-Wesley Pub. Co., Reading, Ma, 1980.

[2] A complete presentation of the mathematical aspects of the variational calculus can be found in I. M. Gelfand and S. V. Fomin, Calculus of Variations, Dover Publications, Mineola, N. Y., 2000.

[3] For an extensive discussion of Lagrangian and Hamiltonian formalisms in relativistic mechanics and field theory, including applications to the electromagnetic field theory, see L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Pergamon Press, Oxford, 1975.

[4] The elementary introduction of the energy-momentum tensor is based on I. R. Kenyon, General Relativity, Oxford University Press, Oxford, 1990.