PHYS 3033 GENERAL RELATIVITY PART II
Chapter 5
The physical basis of the general theory of relativity

> "I was sitting in a chair in the patent office at Berne, when all of a sudden a thought occurred to me: *if a person falls, he will not feel his own weight*"
> Albert Einstein
> Years later, Einstein was to say that this was "*the happiest thought of my life*"

5.1 The concept of mass

Mass arises naturally from Newton's second law of motion. For a **given body** we can measure its **acceleration** (assuming only that we can readily measure length and time) and hence **the force** acting on it in terms of its mass. The relation between mass and force is given by $\vec{F} = m\ddot{\vec{r}}$.

Although it is quite usual to refer to the scalar coefficient m in this relation as the mass, strictly speaking, **what we detect when we apply a force to a body is its inertia**.

The second approach to a more quantitative definition of the mass of a body is through Newton's law of universal gravitation. This law states that the gravitational force between two bodies of mass m and M separated by a distance r is $F = -G\frac{mM}{r^2}$.

Therefore according to Newton's second law the equation of motion of the particle of mass m in the field created by the particle of mass M is

$$m\frac{d^2r}{dt^2} = -G\frac{mM}{r^2}. \tag{1}$$

We must now ask: **is it correct to cancel the mass m in Eq. (1)?** Are the m's in both sides really the same? **The mass that appears on the left side of Eq. (1) is the inertial mass.**

For any arbitrary body, the inertial mass is defined as follows[1]: we take the body and let it **interact,** somehow, with a **standard inertial mass** (one kilogram).

Both the body and the standard will **accelerate** towards or away from one another. Designating the acceleration of the body and of the standard by a and a_s, respectively, we can define the inertial mass by

$$m_i = \frac{a_s}{a}.$$

The **gravitational mass** can be defined in a similar manner. We take a standard object and define its **gravitational mass to be one unit**; for convenience we will use the **standard kilogram as a standard for both inertial and gravitational mass**.

We now place the body at some distance r and let it **interact gravitationally** with the standard. The body will accelerate towards the standard. We define the gravitational mass of the body in terms of this acceleration and the distance between the objects:

$$m_G \equiv \lim_{r\to\infty} \frac{am_i r^2}{G}.$$

Alternatively, using $m_i = \dfrac{a_s}{a}$ we can write this as

$$m_G \equiv \lim_{r\to\infty} \frac{a_s r^2}{G}.$$

This gives the gravitational mass in kilograms.

The limiting procedure $r \to \infty$ is needed in order to eliminate the effects of multipole fields, which depend on the mass distribution of the two bodies. Also, proceeding to the limit $r \to \infty$ eliminates the effect of shorter range forces (nuclear force, Van der Waals force etc.).

At large distances only the gravitational and electrostatic forces will remain, but the latter can be eliminated by taking the precaution of keeping the **standard body neutral.**

If we take two identical copies of the standard mass and let them fall towards each other, the acceleration of each serves to define the constant G:

$$G = \lim_{r\to\infty}\left(a_{ss} r^2\right).$$

With these precise definitions of m_i and m_G it is clear that the gravitational force between two particles is

$$F = -G\frac{m_G M_G}{r^2},$$

and the equation of motion is

$$m_i \frac{d^2 r}{dt^2} = -G\frac{m_G M_G}{r^2}.$$

Whether all particles fall in the gravitational field of the particle of mass M_G with the same acceleration depends on whether all particles have the same value of $\dfrac{m_i}{m_G}$.

If this ratio is **a universal constant**, it must have the value one (the standard body has this value by definition). The question is then, is the equation

$$m_i = m_G \ ,$$

satisfied for all bodies?

This is a question which can be answered only by experimental means.

5.2 The principle of equivalence

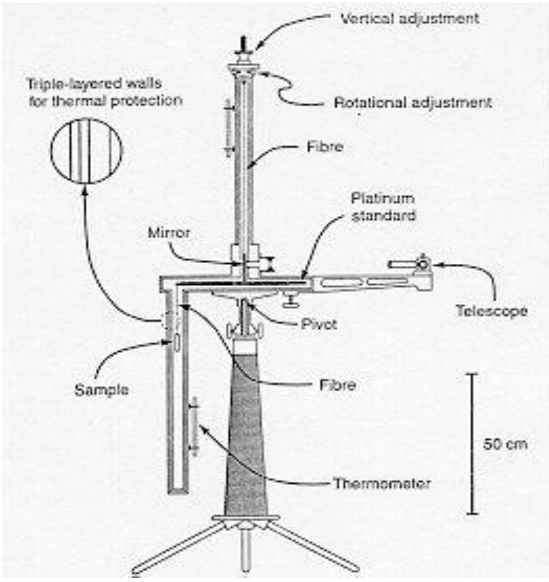
At the end of the 16th century, Galileo Galilei demonstrated that **the speed of falling bodies is not proportional to their weight**, throwing two spheres (one of lead and one of cork, the former being more than one hundred times as heavy as the latter) from the leaning tower of Pisa: the surprised audience had to admit that in spite of their different weights**, the two objects with different masses reached the ground at the same time**. Four centuries later, this experiment would still disconcert people.

The first careful experiments specifically designed to test the equality of inertial and gravitational mass are due to Newton.

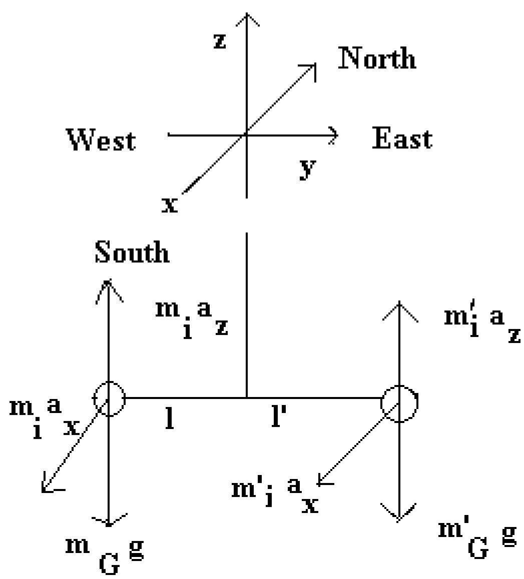
The most precise results have been obtained with **torsion balances**, which have been used for the first time to check the equality of the inertial and gravitational mass at a very high level of precision by Eotvos.

L .Eotvos (1849-1919)

Lóránd Eötvös (1848-1919) studied at Heidelberg and received a doctorate with a thesis which studied problems proposed by Fizeau on the relative motion of a light source. This was one of the first steps towards relativity theory. Eötvös went back to Hungary in 1871. He taught at the University of Budapest from 1871 and he became professor of experimental physics in 1878. He published on capillarity between 1876 and 1886. For the rest of his life he published on gravitation He invented the Eötvös balance and showed that, to a high degree of accuracy, gravitational mass and inertial mass are equivalent. Eotvos work was highly praised by Einstein, who wrote: "**Eotvos is the last classic of the classical physics.**"



In the Eotvos experiment [2] two pieces of matter, labeled weight, are attached to the arms of a torsion balance. These weights are made of different substances, e. g. platinum and copper. If m_i / m_G has a different value for the two substances, then a torque will be exerted on the balance. The forces are of two kinds: there is **the gravitational force** $m_G \vec{g}$ **exerted by the Earth** and the **centrifugal pseudo-force** $m_i \vec{a}$ **produced by the rotation of the Earth**; the quantity $\vec{g}$ is the acceleration of gravity and $\vec{a}$ is the centrifugal acceleration due to Earth rotation at the locality of the experiment.



The beam of the balance points in the east-west direction: the centrifugal force has a vertical (opposite to $\vec{g}$ component) $m_i a_z$ and a horizontal component $m_i a_x$.
The torque about the $z-$axis is

$$\tau = m_i a_x l - m_i' a_x l'.$$

We can eliminate l' by using the equilibrium condition for rotation about the x-axis,

$$(m_G g - m_i a_z) l = (m_G' g - m_i' a_z) l',$$

and obtain

$$\tau = m_i a_x l \left(1 - \frac{m_i^{'}}{m_i} \frac{m_G g - m_i a_z}{m_G^{'} g - m_i^{'} a_z} \right) = m_i a_x l \frac{g\left(\frac{m_G^{'}}{m_i^{'}} - \frac{m_G}{m_i} \right)}{g \frac{m_G^{'}}{m_i^{'}} - a_z}.$$

From this it is obvious that a torque exists only if $\frac{m_G^{'}}{m_i^{'}} \neq \frac{m_G}{m_i}$. In equilibrium, the gravitational torque is compensated by a torque produced by the suspension fiber.

The presence of the gravitational torque can be detected by rotating the entire apparatus by exactly 180° **about the vertical axis**. If the equilibrium position of the beam was exactly along the east-west direction before this rotation, then it will be slightly off after the rotation**: this change in equilibrium position occurs because turning the apparatus around changes the sign of the torque**.

Eotvos's experiment has shown that

$$\frac{|m_i - m_G|}{m_i} < 3 \times 10^{-9}.$$

Many other very precise experiments have tested the ratio m_i / m_G for a wide variety of materials, with a precision as high as 10^{-13}.

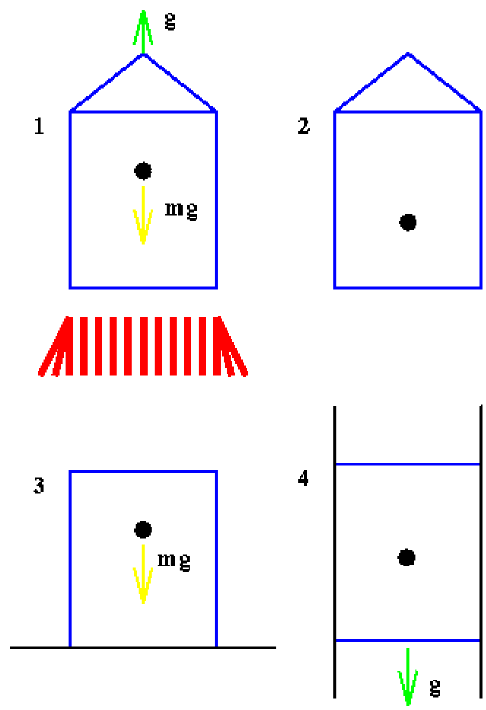
If we go further and assert that the two kinds of mass are actually the same, then we can formulate the following formal statement, called the **Principle of equivalence**:

The principle of equivalence states that inertial mass equals gravitational mass.

5.3 Einstein's lift experiment; the relativistic equivalence principle

The equality of the inertial and gravitational mass led Einstein to suggest that gravity is in some sense an inertial force. That is, **he postulated that the gravitational force was an effect which arose from the use of a non-inertial frame.**

In a paper published in 1914 Einstein discussed a series of thought experiments, which are sometimes referred as **the lift experiments**. As these experiments are no longer thought experiments, having actually being carried out, we shall think of an experimenter in a **rocket ship**, rather than in a lift[3]. The essential feature is that the **experimenter can only observe the behavior of a test mass inside the rocket ship**. In each of these cases we take the **inertial frame S** to have its origin at the centre of the Earth and its acceleration determined by the fixed stars. When **the rocket is accelerating its frame is denoted by** $S_a^{'}$ and when it is **moving at a constant speed with respect to S it is denoted S'.**

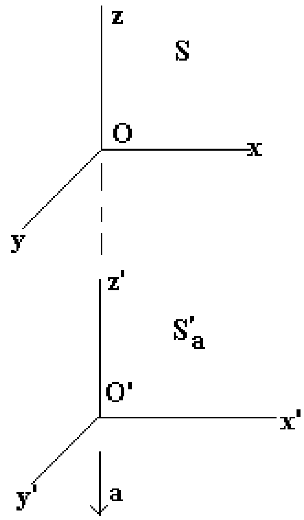


We begin with the rocket before take-off, when it is still **at rest on the Earth. The experiment consists of releasing the test mass and allowing it to fall to the floor. The experimenter measures its acceleration and finds to be g, the local acceleration due to gravity.** Now the rocket takes off. At some later time it is a long way from the Earth, and **is moving at a constant speed relative to S. Under these circumstances, when the observer releases the rest mass, it remains where it was released.** In both S and S' the rest mass has no net forces acting on it. In S it moves uniformly with S', while in S' it is at rest.

Next we consider the case where the rocket motors have been switched on, and **it accelerates at a uniform rate** a. It is still a long way from the Earth, so that Earth's gravity can still be neglected. To an observer in S **the floor of the ship accelerates towards the test mass when the experimenter releases it**. However, to the observer in $S_a^{'}$ **the mass appears to accelerate towards the floor**, as if under the influence of a force of magnitude ma.

Lastly, we imagine that the rocket is now in **free fall back to Earth** and is close enough for the Earth' gravitational field to affect it. To an observer in the Earth's frame S, the rocket, experimenter and test mass **all appear to be falling to Earth with acceleration** g**.** However, **to the experimenter in the rocket the test mass remains motionless and he concludes that no forces are acting on it.**

Let the non-inertial frame $S_a^{'}$ move with constant acceleration a in the direction



of the negative $z-$axis of S.

The two coordinates at right angles to the motion are of course unaffected and thus

$$x = x',\ y = y'.$$

Assuming for simplicity that $t = t'$, the transformations in the z direction are

$$z = z' - \frac{at^2}{2},$$

$$\frac{dz}{dt} = \frac{dz'}{dt} - at$$

and

$$\frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2} - a.$$

In S Newton's second law is just

$$m\frac{d^2z}{dt^2} = F.$$

In $S_a^{'}$, using the acceleration transformation we obtain Newton's second law in the form

$$m\frac{d^2z'}{dt^2} = F + ma.$$

If the experimental apparatus (a box, say) which defines $S_a^{'}$ is in free fall towards the Earth, then the gravitational force inside the box is cancelled by the inertial force ma. That is

$$m\frac{d^2z'}{dt^2} = -mg + ma\,,$$

and $a = g$ for $S_a^{'}$ in free fall. Therefore a particle in a box experiences no force (or acceleration) relative to the box. However, **if we move the box to some region of the universe where there is no gravitational force, and accelerate it with $a = g$, then every particle in the box will experience an apparent force of magnitude mg in the direction of $-z$.**

The overall conclusion from these experiments must be that **the experimenter in the rocket is unable to tell whether forces acting on the test mass are due to gravity or to some inertial forces. The observer in S can tell**, but she/he is in a privileged position, being completely outside the non-inertial frame which is under consideration.

Considerations of this kind led Einstein to the idea that perhaps **gravity was also an inertial force.**

Einstein's conclusion was formulated as the principle of equivalence.

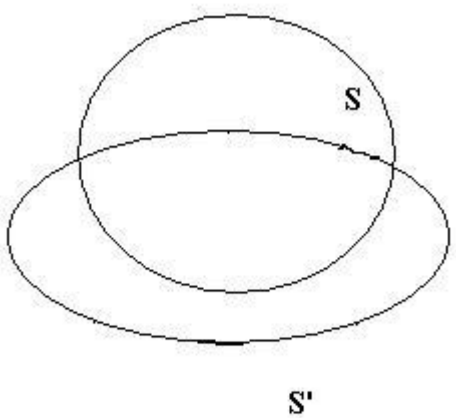
The Principle of equivalence states that a frame undergoing constant acceleration is locally indistinguishable from a frame at rest or in uniform motion in a gravitational field.

This is known as the **weak form of the principle of equivalence**. We restrict the acceleration to constant acceleration and a region to a local region over which the gravitational effects have no spatial inhomogeneity.

The principle of equivalence can also be restated as

No local experiment can distinguish between the free fall of a body in a gravitational field and the uniform motion of the same body in the absence of a gravitational field.

5.4 The necessity for a curved metric



Let us consider two reference frames, of which one **(S) is inertial,** while the other **(S') rotates uniformly with respect to S around their common z axis**. A circle in the x, y plane of the S system (with its center at the origin) can also be regarded as a circle in the x', y' plane of the S' system. **Measuring the length of the circle and its diameter with a yardstick in S system we obtain values whose ratio is π,** in accordance to the Euclidian character of the geometry in the inertial reference system. Now let the measurement be carried out with a yardstick at rest relative to S'. **Observing this process from the S system, we find that the yardstick laid along the circumference suffers a Lorentz contraction, whereas the yardstick placed radially is not changed**. It is therefore clear that **the ratio of the circumference to the diameter, obtained from such a measurement will be greater than π.**

In an inertial reference system, in Cartesian coordinates, the interval ds is given by the relation

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

Upon transforming to any other inertial reference system (i.e. under Lorentz transformations), the interval retains its form. However, **if we transform to a non-inertial frame of reference, ds^2 will be no longer the sum of squares of the four coordinate differentials.**

Hence, for example, when we transform to a uniformly rotating system of coordinates,

$$x = x' \cos \omega t - y' \sin \omega t , \ \ y = x' \sin \omega t + y' \cos \omega t , \ \ z = z' ,$$

where ω is the angular velocity of the rotation, directed along the z axis,
the interval takes on the form

$$ds^2 = \left[c^2 - \omega^2 \left(x'^2 + y'^2\right)\right] dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\omega y' dx' dt - 2\omega x' dy' dt .$$

No matter what the law of transformation of the time coordinate is, **this expression cannot be represented as a sum of squares of the differentials of the coordinates.**

Thus in a non-inertial system of reference **the square of an interval appears as a quadratic form of general type in the coordinate differential** [4] , that is, it has the form

$$ds^2 = g_{ik} dx^i dx^k , \qquad (1)$$

where the g_{ik} are certain functions of the space coordinates x^1, x^2, x^3 and of the time coordinate x^0.

Thus, when we use a non-inertial system, the four-dimensional coordinate system $\left(x^0, x^1, x^2, x^3\right)$ is curvilinear. **The quantities g_{ik} determining all the geometric properties in each curvilinear system of coordinates represent the space-time metric.**

The quantities g_{ik} can clearly always be considered symmetric in the indices i and k,

$$g_{ik} = g_{ki} ,$$

since they are determined from the symmetric form (1), where g_{ik} and g_{ki} enter as factors of one and the same product $dx^i dx^k$. In the general case there are ten different quantities g_{ik}, four with equal and 6 with different indices. In an inertial reference system with coordinates $\left(x^0 = ct, x^1 = x, x^2 = y, x^3 = z\right)$, the quantities g_{ik} are

$$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1,\ g_{ik} = 0 \ \text{for}\ i \neq k .$$

Such a four-dimensional system of coordinates is also called Galilean.

Any gravitational field is just a change in the metric of space-time, as determined by the quantities g_{ik}.

This important fact means that the geometrical properties of the space-time (its metric) are determined by physical phenomena and are not fixed properties of space and time.

The theory of gravitational fields, constructed on the basis of the theory of relativity, is called the general theory of relativity.

In the general case of an arbitrary, varying gravitational field, the metric of the space is not only non-Euclidian, but also varies with time. This means that the relation between different geometrical distances change with time. As a result, **the relative position of test bodies introduced into the field cannot remain unchanged in any coordinate system**. Thus in the general theory of relativity it is impossible in general to have a system of bodies which are fixed relative to each other.

In connection **with the arbitrariness of the choice of a reference system, the laws of nature must be written in the general theory of relativity in a form which is**

appropriate to any four-dimensional system of coordinates. In other words the laws of nature must be written **in a covariant form** [5]. This, of course, does not imply the physical equivalence of all these reference systems (like the physical equivalence of all inertial reference frames in the special theory of relativity). On the contrary, the specific appearances of physical phenomena, including the properties of the motion of the bodies, become different in all systems of reference.

Notes

[1]The definition of the inertial and gravitational mass follows the definitions in H. C. Ohanian, Gravitation and spacetime, W. W. Norton and Company Inc., New York, 1976.

[2] The discussion of the Eotvos experiment is based on H. C. Ohanian, Gravitation and spacetime, W. W. Norton and Company Inc., New York, 1976; other experiments testing the equality of the inertial and gravitational mass are described in I. R. Kenyon, General Relativity, Oxford University Press, Oxford, 1990 and S. Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity New York, Wiley, 1972.

[3] The rocket thought experiment is also discussed in I. R. Kenyon, General Relativity, Oxford University Press, Oxford, 1990.

[4] For the necessity of the introduction of a curved metric see also S. Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity New York, Wiley, 1972; L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971 and Ya. B. Zeldovich and I.D. Novikov, Stars and relativity, Mineola, N.Y., Dover Publications, 1996.

[5] Critical discussions of the physical basis of the general theory of relativity are given in V. A. Fock, The theory of space, time and gravitation, New York, Pergamon Press, 1959 and R. P. Feynman, Lectures on gravitation, Pasadena, California Institute of Technology, 1971.