PHYS 3033 GENERAL RELATIVITY PART II
Chapter 6
Riemannian geometry I

"…geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child...."

William Clifford (1845-1879)

6.1 Vectors and tensors

Since in studying the gravitational fields we are confronted with the necessity of considering phenomena in **an arbitrary reference frame**, it is necessary to develop **four-dimensional geometry in arbitrary curvilinear coordinates**[1] **.**

Let us consider the transformation from one coordinate system $\left(x^0, x^1, x^2, x^3\right)$ to another $\left(x'^0, x'^1, x'^2, x'^3\right)$,

$$x^i = f^i\left(x'^0, x'^1, x'^2, x'^3\right).$$

Here f^i are certain general arbitrary functions satisfying some regularity conditions.

The condition **that the** x^i **'s are independent demands that the Jacobian of the transformation does not vanish**, that is

$$J \equiv \begin{vmatrix} \dfrac{\partial x^0}{\partial x'^0} & \dfrac{\partial x^0}{\partial x'^1} & \dfrac{\partial x^0}{\partial x'^2} & \dfrac{\partial x^0}{\partial x'^3} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial x^3}{\partial x'^0} & \dfrac{\partial x^3}{\partial x'^1} & \dfrac{\partial x^3}{\partial x'^2} & \dfrac{\partial x^3}{\partial x'^3} \end{vmatrix} \neq 0 .$$

When we transform the coordinates, their differentials transform according to the relation

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k .$$

A set of four quantities $A^i \left(i = 0,1,2,3\right)$**, which under a general transformation of the coordinates transform like the coordinates differentials is called a contravariant four-vector:**

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k . \tag{1}$$

Let ϕ a scalar function. Under a coordinates transformation the four quantities $\partial\phi/\partial x^i$ transform according to the formula

$$\frac{\partial\phi}{\partial x^i}=\frac{\partial\phi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i},$$

which is different from formula (1).

A set of four quantities $A_i(i=0,1,2,3)$ which under a general coordinates transformation transform like the derivative of a scalar is called a covariant four-vector:

$$A_i=\frac{\partial x'^k}{\partial x^i}A_k'.$$

Because two types of vectors appear in curvilinear coordinates, there are three types of tensors of second rank. **We define a contravariant tensor of the second rank A^{ik} a set of sixteen quantities which transform like the products of the components of two contravariant vectors,** i. e. according the law:

$$A^{ik}=\frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m}A'^{lm}.$$

A covariant tensor of rank two transforms according to the expression

$$A_{ik}=\frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k}A_{lm}'.$$

A mixed tensor transforms as

$$A_k^i=\frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^k}A_m'^l.$$

The definitions given here are the natural generalizations of the definitions of the four-vectors and four tensors in Galilean (Minkowskian) coordinates, according to which the **differentials dx^i form a contravariant four-vector** and **the derivatives $\partial\phi/\partial x^i$ form a covariant four-vector**. The definitions given above can be generalized to define tensors of arbitrary order [2].

The rules for forming four-tensors by multiplication or contraction of products of other four-tensors remain the same in curvilinear coordinates as they were in Galilean coordinates. It is easy to see that **the product of two four-vectors is a scalar**, invariant with respect to the coordinate transformation:

$$A^i B_i = \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^i} A'^l B'_m = \frac{\partial x'^m}{\partial x'^l} A'^l B'_m = A'^l B'_l,$$

where we have used the property of the partial derivatives of the coordinates given by $\frac{\partial x'^m}{\partial x'^l} = \delta_l^m$.

The **unit four-tensor** δ_i^k is defined again in the form

$$\delta_i^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}.$$

If A^k is a four-vector, then multiplying by δ_k^i we obtain

$$A^k \delta_k^i = A^i,$$

that is another four-vector; **this proves that δ_i^k is a tensor**.

The square of the line element (interval) in curvilinear coordinates is a quadratic form in the differentials dx^i :

$$ds^2 = g_{ik} dx^i dx^k,$$

where the g_{ik} are functions of the coordinates: g_{ik} is symmetric in the indices i and k :

$$g_{ik} = g_{ki}.$$

Since the contracted product of g_{ik} and the contravariant tensor $dx^i dx^k$ is a scalar, it follows **that the g_{ik} 's form a covariant tensor; it is called the metric tensor.**

To tensors are said to be reciprocal to each other if

$$A_{ik} B^{kl} = \delta_i^l.$$

In particular the contravariant metric tensor is the tensor g^{ik} reciprocal to the tensor g_{ik} :

$$g_{ik} g^{kl} = \delta_i^l.$$

The same physical quantities can be represented in contra or covariant components. It is obvious that **the only quantities that can determine the connection between the different forms are the components of the metric tensor**. This connection is given by the formulae:

$$A^i = g^{ik} A_k , \; A_i = g_{ik} A^k .$$

The transition between the different forms of a given physical tensor is accomplished by using the metric tensor according to the formulas:

$$A^i_k = g^{il} A_{lk} , A^{ik} = g^{il} g^{km} A_{lm} \text{ etc.}$$

In the previous lectures we have defined **the completely antisymmetric unit pseudotensor** e^{iklm}, with $e^{0123} = 1$ and $e_{0123} = -1$. Let us transform this tensor to an arbitrary system of coordinates and denote it by E^{iklm}. Let the x'^i be Galilean and the x^i be arbitrary curvilinear coordinates. According to the general rules for transformation of tensors we have

$$E^{iklm} = \frac{\partial x^i}{\partial x'^p} \frac{\partial x^k}{\partial x'^r} \frac{\partial x^l}{\partial x'^s} \frac{\partial x^m}{\partial x'^t} e^{prst} ,$$

or

$$E^{iklm} = J e^{iklm} ,$$

where J is the Jacobian of the transformation.

The Jacobian can be expressed in terms of the determinant of the metric tensor g_{ik} (in the system x^i). To do this we write the formula for the transformation of the metric tensor:

$$g^{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} g^{lm(0)} ,$$

and equate the determinants of the two sides of this equation. The determinant of the reciprocal tensor is

$$\left| g^{ik} \right| = 1 / \left| g_{ik} \right| = 1/g .$$

The determinant $\left| g^{lm(0)} \right| = -1$. Thus we have

$$\frac{1}{g} = -J^2 .$$

Thus in curvilinear coordinates the antisymmetric unit tensor of rank four must be defined as

$$E^{iklm} = \frac{1}{\sqrt{-g}} e^{iklm} .$$

The indices of this tensor are lowered by using the formula

$$e^{prst} g_{ip} g_{kr} g_{ls} g_{mt} = -g e_{iklm} ,$$

so that its covariant components are

$$E_{iklm} = \sqrt{-g} e_{iklm} .$$

In a Galilean coordinate system x'^i the integral of a scalar with respect to $d\Omega' = dx'^0\, dx'^1\, dx'^2\, dx'^3$ is also a scalar. On transforming to curvilinear coordinates x^i, the element of integration goes over into

$$d\Omega' \to \frac{1}{J} d\Omega = \sqrt{-g}\, d\Omega .$$

Thus in curvilinear coordinates, when integrating over a four-volume the quantity $\sqrt{-g}\, d\Omega$ behaves like an invariant.

The element of area of the hypersurface spanned by three infinitesimal displacements is the contravariant antisymmetric tensor dS^{ikl}; the vector dual to it is obtained by multiplying by the tensor $\sqrt{-g} e_{iklm}$, so it is equal to

$$\sqrt{-g}\, dS_i = -\frac{1}{6} e_{iklm} dS^{klm} \sqrt{-g} .$$

Similarly, if df^{ik} is the element of the two-dimensional surface spanned by two infinitesimal displacements, the dual tensor is defined as

$$\sqrt{-g}\, d^*_{ik} = \frac{1}{2} \sqrt{-g} e_{iklm} df^{lm} .$$

The rules for transforming the various integrals into one another remain the same. Of particular importance is the rule for transforming the integral over a hypersurface into an integral over a four-volume (Gauss theorem), which is accomplished by the substitution:

$$dS_i \to d\Omega \frac{\partial}{\partial x^i} .$$

6.2 Covariant differentiation

In Galilean coordinates the differentials dA_i of a vector form a vector and the derivatives $\partial A_i / \partial x^k$ of the components of a vector with respect to the coordinates form a tensor.

In curvilinear coordinates this is not so; dA_i **is not a vector and** $\partial A_i / \partial x^k$ **is not a tensor.** This is due to the fact that dA_i **is the difference of vectors located at different (infinitesimally separated) points of space; at different points in space vectors transform differently**, since the coefficients in the transformation formulas are functions of the coordinates.

It is easy to verify this directly. To do this we determine the transformation formulas for the differentials dA_i in curvilinear coordinates. A covariant vector is transformed according to the formula

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k ,$$

therefore

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\frac{\partial x'^k}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l .$$

Thus dA_i does not transform at all like a vector (the same also applies to the differential of a contravariant vector).

Only if the second derivatives $\frac{\partial^2 x'^k}{\partial x^i \partial x^l} = 0$, i. e. if the x'^k are linear functions of the x^k, do the transformation formulas have the form

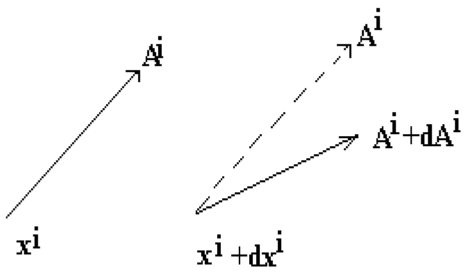
$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k ,$$

that is dA_i transforms like a vector.

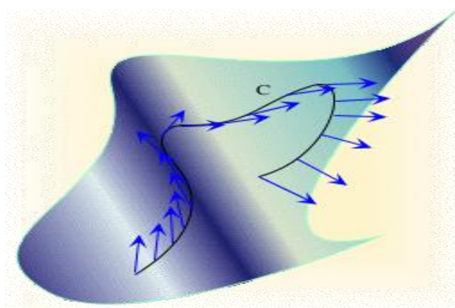
We now undertake the definition of a tensor which in curvilinear coordinates plays the same role as $\partial A_i / \partial x^k$ in Galilean coordinates. In other words we must transform $\partial A_i / \partial x^k$ from Galilean to curvilinear coordinates.

In curvilinear coordinates, in order to obtain a differential of a vector which behaves like a vector, it is necessary that the two vectors to be subtracted from each other be located at the same point in space.

In other words we must somehow translate one of the vectors (which are separated infinitesimally from each other) to the point where the second is located, after which we determine the difference of the two vectors which now refer to one and the same point in space.



The operation of translation itself must be **defined so that in Galilean coordinates the difference will coincide with the ordinary differential** dA_i. Since dA_i is just the difference of the components of two infinitesimally separated vectors, this means that when we use Galilean coordinates the components of a vector should not change as a result of the translation operation. But such a translation is precisely the translation of a vector parallel to itself. **Under a parallel translation of a vector its components in Galilean coordinates do not change.**



If, on the other hand, we use curvilinear coordinates, then in general the components of a vector will change under such a translation. Therefore in curvilinear coordinates the difference in the components of two vectors after translating one of them to the point where the other is located will not coincide with their difference before the translation (i.e. with the differential dA_i**).**

Thus to compare two infinitesimally separated vectors we must subject one of them to a parallel translation to the point where the second is located.

Let us consider an arbitrary contravariant vector; if its value at the point x^i is A^i, then at the neighboring point x^i+dx^i it is equal to A^i+dA^i.

We subject the vector A^i to an infinitesimal parallel displacement to the point x^i+dx^i; the change in the vector which results from this we note by δA^i. Then the difference DA^i between the two vectors which are now located at the same point is

$$DA^i = dA^i - \delta A^i. \qquad (1)$$

The change δA^i **in the components of a vector under an infinitesimal parallel displacement depends on the values of the components themselves, where the dependence must clearly be linear.** This follows directly from the fact that the sum of two vectors must transform according to the same law as each of the constituents. Thus δA^i has the form

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \qquad (2)$$

where the Γ^i_{kl} are certain functions of the coordinates. Their form depends, of course, on the coordinate system; **for a Galilean coordinate system $\Gamma^i_{kl} = 0$.**

From this we see that **the quantities Γ^i_{kl} do not form a tensor,** since a **tensor which is equal to zero in one coordinate system is equal to zero in every other one**. In a curvilinear space it is impossible to make all the Γ^i_{kl} to vanish over all space.

But **the principle of equivalence requires that by a suitable choice of coordinate system we can eliminate the gravitational field over a given infinitesimal region of space, i.e. we can make the quantities Γ^i_{kl} vanish in it.**

The quantities Γ^i_{kl} are called connection coefficients or Christoffel symbols.

E. B. Christoffel (1829-1900)

Christoffel studied at the University of Berlin from 1850. In 1856 a doctorate was awarded to him for a dissertation on the motion of electricity in homogeneous bodies. He wrote important papers which contributed to the development of the tensor calculus. The Christoffel symbols, which he introduced, are fundamental in the study of tensor analysis. This allowed Ricci and Levi-Civita to develop a coordinate free differential calculus which Einstein, with the help of Grossmann, turned into the tensor analysis mathematical foundation of general relativity.

In addition to the quantities Γ^i_{kl} we shall also use the quantities $\Gamma_{i,kl}$ defined as follows

$$\Gamma_{i,kl} = g_{im}\Gamma^m_{kl}.$$

Conversely,

$$\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}.$$

It is easy to relate the change in the components of a covariant vector under a parallel displacement to the Christoffel symbols. To do this we note that under a parallel displacement a scalar is unchanged. In particular, **the scalar product of two vectors does not change under a parallel displacement.**

Let A^i and B^i be any covariant and contravariant vectors. Then from $\delta\left(A_i B^i\right) = 0$ we have

$$\delta\left(A_i B^i\right) = \delta A_i B^i + A_i \delta B^i = 0,$$

$$\delta A_i B^i = -A_i \delta B^i = \Gamma^i_{kl} B^k A_i dx^l,$$

or

$$B^i \delta A_i = \Gamma^k_{il} B^i A_k dx^l .$$

From this, in view of the arbitrariness of the B^i we obtain

$$\delta A_i = \Gamma^k_{il} A_k dx^l ,$$

which determines the change in a covariant vector under a parallel displacement.

Substituting (2) and $dA^i = \left(\partial A^i / \partial x^l\right) dx^l$ in (1) we obtain

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k \right) dx^l .$$

Similarly, for the covariant vector we find

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k \right) dx^l .$$

The expressions in the parentheses are tensors, since when multiplied by the vector dx^l they give a vector.

These tensors give the generalization of the concept of a derivative to curvilinear coordinates. They are called the covariant derivatives of the vectors A^i **and** A_i **, respectively.** We shall denote them by $A^i_{;k}$ and $A_{i;k}$. Thus

$$DA^i = A^i_{;l} dx^l , \; DA_i = A_{i;l} dx^l ,$$

while the covariant derivatives themselves are

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k , \; A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k .$$

In Galilean coordinates $\Gamma^i_{kl} \equiv 0$ **and covariant differentiation reduces to ordinary differentiation**.

One can similarly determine **the covariant derivative of a tensor of arbitrary rank.** In doing this one finds the following rule of covariant differentiation:

-to obtain the covariant derivative of a tensor $A^{\cdot\cdot}_{\dots}$ with respect to x^i , we add to the ordinary derivative $\partial A^{\cdot\cdot}_{\dots} / \partial x^i$ for each covariant index i ($A^{\cdot\cdot}_{.i.}$) a term $-\Gamma^k_{il} A^{\cdot\cdot}_{.k.}$ and for each contravariant index i ($A^{\cdot k \cdot}_{\dots}$) a term $+\Gamma^i_{kl} A^{\cdot k \cdot}_{\dots}$.

Exercise. Find the covariant derivative of the metric tensor g_{ik}.

One can easily verify that the covariant derivative of a product is found by the same rule as for ordinary differentiation in general. In doing this we must consider the covariant derivative of a scalar ϕ as an ordinary derivative, that is, as the covariant vector $\phi_k = \partial\phi / \partial x^k$, in accordance with the fact that for a scalar $\delta\phi = 0$. Hence, the covariant derivative of a product $A_i B_k$ is

$$\left(A_i B_k\right)_{;l} = A_{i;l} B_k + A_i B_{k;l}.$$

If in a covariant derivative we raise the index signifying the differentiation, we obtain the so-called contravariant derivative. Thus

$$A_i^{;k} = g^{kl} A_{i;l}, \; A^{i;k} = g^{kl} A^i_{;l}.$$

The formulae for the transformation of the Christoffel symbols can be derived by comparing the two equations that determine the covariant derivatives and requiring that these laws be the same for both. A simple calculation gives:

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m}. \tag{3}$$

Exercise. Find the law of transformation of the Christoffel symbols.

From this formula we see that the Γ^i_{kl} transforms like a tensor only for linear coordinates (when the second term in the equation drops out).

However, we note that this term is symmetric in k and l, and therefore drops out for the transformation of $S^i_{kl} = \Gamma^i_{kl} - \Gamma^i_{lk}$. This quantity therefore transforms like a tensor

$$S^i_{kl} = S'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l}.$$

S^i_{kl} is called the torsion tensor of the space.

We now show that in the gravitational theory based on the equivalence principle the torsion tensor must be zero. By virtue of the equivalence principle there must be a Galilean coordinate system in which the Γ^i_{kl} and consequently also the S^i_{kl} vanish at a given point. Since S^i_{kl} is a tensor, **if it vanishes in one coordinate system it must vanish in all frames.** This means that the Christoffel symbols must be symmetric in their lower indices:

$$\Gamma^i_{kl} = \Gamma^i_{lk}, \; \Gamma_{i,kl} = \Gamma_{i,lk}.$$

There are generally 40 different quantities Γ^i_{kl}.

The transformation formula and the symmetry of the Christoffel symbols enables to prove the important result that **it is always possible to choose a coordinate system in which all the Γ^i_{kl} become zero at a previously assigned point (such a system is called locally inertial or locally geodesic).**

Let the given point be chosen as the origin of coordinates, and let the values of the Γ^i_{kl} at it be initially (in the coordinates x^i) equal to $\left(\Gamma^i_{kl}\right)_0$. In the neighborhood of this point we make the transformation

$$x'^i = x^i + \frac{1}{2}\left(\Gamma^i_{kl}\right)_0 x^k x^l . \tag{4}$$

Then

$$\left(\frac{\partial^2 x'^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial x'^m}\right)_0 = \left(\Gamma^i_{kl}\right)_0 ,$$

and according to the transformation law of the Christoffel symbols all the Γ'^m_{np} become equal to zero.

For the transformation (4), $\left(\partial x'^i / \partial x^k\right)_0 = \delta^i_k$, so that it does not change the value of any tensor (including the metric tensor) at the given point, so that we can make the Christoffel symbols vanish at the same time as we bring the metric tensor to the Galilean form.

6.3 The relation of the Christoffel symbols to the metric tensor

Let us show now that **the covariant derivative of the metric tensor g_{ik} is zero**[3]**.** To do this we note that the relation

$$DA_i = g_{ik} DA^k$$

is valid for the vector DA_i as for any other vector. On the other hand $A_i = g_{ik}A^k$, so that

$$DA_i = D\left(g_{ik}A^k\right) = g_{ik}DA^k + A^k Dg_{ik} = g_{ik}DA^k .$$

Since the vector A^k is arbitrary, it follows that Dg_{ik} and therefore the covariant derivative satisfies the fundamental relation

$$g_{ik;l} = 0 .$$

This equation can be used to express the Christoffel symbols in terms of the metric tensor g_{ik}. To do this we write first

$$g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,il} - \Gamma_{i,kl} = 0 .$$

Thus the derivatives of the metric tensor are expressed in terms of the Christoffel symbols. By permuting the indices i,k and l we obtain

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl} ,$$
$$\frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik} ,$$
$$-\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li} .$$

Taking half the sum of these equations we find

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right).$$

For the symbols Γ^i_{kl} we have [4]

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

The determinant g of the metric tensor g_{ik} and its differential play an important role in many calculations. The differential dg can be obtained by taking the differential of each component of the metric tensor g_{ik} and multiplying it by the corresponding minor A^{ik},

$$dg = A^{ik} dg_{ik} .$$

Therefore the minor of the element g_{ik} is given by

$$A^{ik} = \frac{\partial g}{\partial g_{ik}}.$$

On the other hand the components of the tensor g^{ik}, reciprocal to g_{ik} are given by the minors of the determinant of the g_{ik} divided by the determinant

$$g^{ik} = \frac{A^{ik}}{g}.$$

Hence for the components of the tensor g^{ik}, the reciprocal to the tensor g_{ik} ($g_{ik}g^{kl} = \delta_i^l$) we find the important relation

$$g^{ik} = \frac{1}{g}\frac{\partial g}{\partial g_{ik}}.$$

Exercise. Find the inverse components of the metric tensor for a metric of the form $ds^2 = g_{00}\left(dx^0\right)^2 - g_{11}\left(dx^1\right)^2 - g_{22}\left(dx^2\right)^2 - g_{33}\left(dx^3\right)^2$.

Exercise. Find the inverse components of the metric tensor for a metric of the form $ds^2 = g_{00}\left(dx^0\right)^2 - g_{11}\left(dx^1\right)^2 - g_{22}\left(dx^2\right)^2 - g_{33}\left(dx^3\right)^2 + g_{01}dx^0dx^1$.

For the differential of the determinant of the metric tensor we have

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik}.$$

To obtain the second expression we have used the equality $g_{ik}g^{ik} = \delta_i^i = 4$, leading to $dg_{ik}g^{ik} = -g_{ik}dg^{ik}$.

Exercise. Show that the contracted Christoffel symbol Γ_{ik}^i is given by $\Gamma_{ik}^i = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial}{\partial x^k}\ln\sqrt{-g}$.

With the aid of the formulas which we have obtained above we can transform the expression for $A_{;i}^i$, the generalized divergence of a vector in curvilinear coordinates in a convenient form. We have

$$A_{;i}^i = \frac{\partial A^i}{\partial x^i} + \Gamma_{li}^l A^l = \frac{\partial A^i}{\partial x^i} + A^l\frac{\partial}{\partial x^l}\ln\sqrt{-g} = \frac{\partial A^i}{\partial x^i} + A^i\frac{\partial}{\partial x^i}\ln\sqrt{-g},$$

or, finally,

$$A_{;i}^i = \frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}A^i\right)}{\partial x^i}.$$

Exercise. Show that the divergence of an antisymmetric tensor A^{ik} is given by $A_{;k}^{ik} = \frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}A^{ik}\right)}{\partial x^k}$.

Suppose that A_{ik} is a symmetric tensor; we want to calculate the expression $A^k_{i;k}$ for its mixed components. We have

$$A^k_{i;k} = \frac{\partial A^k_i}{\partial x^k} + \Gamma^k_{lk} A^l_i - \Gamma^l_{ik} A^k_l = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, A^k_i\right)}{\partial x^k} - \Gamma^l_{ki} A^k_l .$$

The last term equals

$$-\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right) A^{kl} .$$

Because of the symmetry of the tensor A^{kl}, two of the terms in parentheses cancel each other, leaving

$$A^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, A^k_i\right)}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} A^{kl} .$$

Finally, we mention that Gauss's theorem, for the transformation of the integral of a vector over a hypersurface into an integral over a four-volume can be written as

$$\oint A^i \sqrt{-g}\, dS_i = \int A^i_{;i} \sqrt{-g}\, d\Omega .$$

Notes

[1] There are many books devoted to an in depth presentation of the differential geometry. The physicist's approach is developed in S. Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity New York, Wiley, 1972; L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971 and H. C. Ohanian, Gravitation and spacetime, W.W. Norton and Comp., New York, 1976. The mathematician's approach to differential geometry can be found in S. Hassani, Mathematical physics: a modern introduction to its foundations, New York, Springer, 1999, C. J. Isham, Modern differential geometry for physicists, World Scientific, Singapore, New Jersey, London, Hong Kong, 1999 or M. Nakahara, Geometry, topology and physics, Bristol, England, A. Hilger, 1990.

[2] The general definition of vectors and tensors follows L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971.

[3] The proof of the constancy of the metric tensor components under covariant differentiation follows the proof given in L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971.

[4] The Christoffel symbols are introduced in a different way in S. Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity New York, Wiley, 1972, by considering the free-fall motion in arbitrary coordinate systems.