PHYS 3033 GENERAL RELATIVITY PART II
Chapter 7
Space and time in general relativity

> "I wish to show that space-time is not necessarily something to which one can ascribe a separate existence, independent of the actual objects of physical reality. Physical objects are not – *in space* – but these objects are – *spatially extended* -. In this way the concept – *empty space* – loses its meaning."
>
> A. Einstein

7.1 Distances and time intervals

As in non-relativistic mechanics, **in general relativity there is a fundamental difference between actual gravitational fields and fields to which non-inertial reference systems are equivalent**[1]**.** [[1] For a discussion of the relation between the field and geometrical aspects of gravity see V. A. Fock, The theory of space, time and gravitation, New York, Pergamon Press, 1959 and R. P. Feynman, Lectures on gravitation, Pasadena, California Institute of Technology, 1971.] Upon transforming to a non-inertial reference frame the quantities g_{ik} are obtained from their Galilean values by a simple transformation of coordinates, and can be reduced over all space to their Galilean values by the inverse coordinate transformation. That such forms for g_{ik} are very special is clear from the fact that it is impossible by a mere transformation of the four coordinates to bring the ten quantities g_{ik} to a pre-assigned form.

An "actual" gravitational field cannot be eliminated by any transformation of coordinates. In other words, **in the presence of a gravitational field space-time is such that the quantities g_{ik} determining the metric cannot, by any coordinates transformation, be brought to their Galilean values over all space.** Such a space-time is said to be curved, in contrast to the flat space-time, where such a reduction is possible.

By an appropriate choice of coordinates we can, however, bring the quantities g_{ik} to Galilean form at any individual point of the curved (non-Galilean) space-time; **this amount to the reduction to diagonal form of a quadratic form with constant coefficients (the values of g_{ik} at the given point)**. Such a **coordinate system is called Galilean for the given point.**

[Note: The **Galilean transformation** is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. This is the passive transformation point of view. The equations below, although apparently obvious, break down at speeds that approach the speed of light owing to physics described by relativity theory.]

After reduction to diagonal form at the given point, **the matrix of the quantities g_{ik} has one positive and three negative principal values (this set of signs is called the**

signature of the matrix). From this it follows that the determinant g, formed from the quantities g_{ik}, is always negative for a real space-time:

$$g < 0.$$

A change in the metric of the space-time also means a change in the purely spatial metric. **To a Galilean g_{ik} in flat space-time there corresponds a Euclidian geometry of space.** In **a gravitational field, the geometry of space becomes non-Euclidian.** This applies both to true gravitational fields, in which space-time is curved, as well as to fields resulting from the fact that the reference system is non-inertial.

In general relativity the choice of the system of coordinates is not limited in any way; the triplet of space coordinates x^1, x^2, x^3 can be any quantities defining the position of bodies in space, and the time coordinate x^0 can be defined by an arbitrary running clock. **The question arises of how, in terms of the values of the quantities x^0 and x^1, x^2, x^3, we can determine actual distances and time intervals**[2] **.**

First we find the relation between **the proper time**, which from now on we shall denote by τ , to the coordinate x^0.

To do this we consider two infinitesimally separated events, occurring at one and the same point in space. Then the interval ds between the two events is just $cd\tau$, where $d\tau$ is the proper time interval between the two events. Setting $dx^1 = dx^2 = dx^3 = 0$ in the general expression $ds^2 = g_{ik}dx^i dx^k$ we find

$$ds^2 = c^2 d\tau^2 = g_{00}\left(dx^0\right)^2,$$

or

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0.$$

Hence, for the time between any two events occurring at the same point in space we obtain

$$\tau = \frac{1}{c}\int\sqrt{g_{00}}dx^0 .$$

This relation determines the actual time interval (or as it is also called, **the proper time** for the given point in space) for a change of the coordinate x^0. The quantity g_{00}, as we see from these formulae, must be positive:

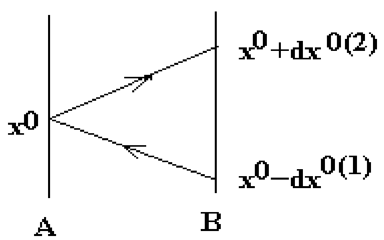
$$g_{00} > 0.$$

Non-fulfillment of this condition means that the corresponding system of reference cannot be realized with real bodies. But if the condition on the principal values is fulfilled, then a suitable transformation of the coordinates can make g_{00} positive.

We now determine the element dl of the spatial distance. In **the special theory of relativity we can define** dl **as the interval between two infinitesimally separated events occurring at one and the same time**. In the general theory of relativity, it is usually impossible to do this, i.e., it is impossible to determine dl by simply setting $dx^0 = 0$ in ds.

This is related to the fact that in a gravitational field the proper time at different points in space has a different dependence on the coordinate x^0.

To find dl we proceed as follows.



Suppose a light signal is directed from some point B in space (with coordinates $x^\alpha + dx^\alpha$) to a point A infinitely near to it, having coordinates x^α, and then back over the same path. Obviously, the time (as observed from the point B) required for this, when multiplied by c, is twice the distance between the two points.

Let us write the interval, separating the space and time coordinates:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}\left(dx^0\right)^2,$$

where it is understood that we sum over repeated Greek indices from 1 to 3. The interval between two events corresponding to the departure and arrival of the signal from one point to the other is equal to zero. [Note; expand $ds^2 = g_{ij}dx^i dx^j$ for i,j varying from 0 to 3]

Solving the equation $ds^2 = 0$ with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\left[-g_{0\alpha}dx^\alpha - \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right],$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left[-g_{0\alpha}dx^\alpha + \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right],$$

corresponding to the propagation of the signal in the two directions between A and B.

If x^0 is the moment of arrival of the signal at A, the times when it left B and when it will return to B are $x^0 + dx^{0(1)}$ and $x^0 + dx^{0(2)}$, respectively. In the diagram the solid lines are the world lines corresponding to the spatial coordinates x^α and $x^\alpha + dx^\alpha$, while the dashed lines are the world lines of the signals.

It is clear that the total interval of "time" between the departure of the signal and its return to the original point is equal to

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}} \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \; .$$

The corresponding interval of proper time is obtained by multiplying by $\sqrt{g_{00}}/c$ and the distance dl between the two points by multiplying once more by $c/2$. As a result, we obtain

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}} \right) dx^\alpha dx^\beta \; .$$

This is the required expression, defining the distance in terms of the space coordinate element. We rewrite it in the form

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{1}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}} \,, \tag{2}$$

is the three-dimensional metric tensor, determining the metric, i.e., the geometric properties of the space. Eqs. (2) give the connection between the metric of the real space and the metric of the four-dimensional space-time.

The g_{ik} generally depend on x^0, so that the space metric (1) also changes with time. **For this reason it is meaningless to integrate** dl; such an integral would depend on the world line chosen between the two given space points.

Thus, in general theory of relativity, the concept of a definite distance between bodies loses its meaning, remaining valid only for infinitesimal distances. The only case where the distance can be defined over a finite domain is that in which the g_{ik} do not depend on the time, so that the integral $\int dl$ along a space curve has a definite meaning.

The tensor $-\gamma_{\alpha\beta}$ is the reciprocal of the contravariant three-dimensional tensor $g^{\alpha\beta}$. From $g^{ik} g_{kl} = \delta^i_l$ we have, in particular,

$$\underline{g^{\alpha\beta} g_{\beta\gamma} + g^{\alpha 0} g_{0\gamma} = \delta^\alpha_\gamma} \; ; \; \underline{g^{\alpha\beta} g_{\beta 0} + g^{\alpha 0} g_{00} = 0} \; ; \; \underline{g^{0\beta} g_{\beta 0} + g^{00} g_{00} = 1} .$$

[Hint $\delta^\alpha_\beta = 0 \, if \, \alpha \neq \beta, and if \, \alpha = \beta, it is 1$]

Determining $g^{\alpha 0}$ from the second equation and substituting in the first we obtain

$$g^{\alpha\beta}g_{\beta\gamma} = \frac{g_{0\gamma}g^{\alpha\beta}g_{0\beta}}{g_{00}} + \delta_{\gamma}^{\alpha}$$ (edited original MS)

$$g^{\alpha\beta}[g_{\beta\gamma} - \frac{g_{0\gamma}g_{0\beta}}{g_{00}}] = \delta_{\gamma}^{\alpha}$$

Which gives

$$-g^{\alpha\beta}\gamma_{\beta\gamma} = \delta_{\gamma}^{\alpha}.$$ ....(Imp) (2.1a)

This result can be formulated differently, by the statement that the quantities $-g^{\alpha\beta}$ form the contravariant three-dimensional metric tensor corresponding to the metric (2): [$-g^{\alpha\beta} = \delta_{\gamma}^{\alpha} / \gamma_{\beta\gamma} = \delta_{\gamma}^{\alpha}\gamma^{\beta\gamma} = \gamma^{\beta\alpha}$ hence]

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}. \tag{2.2}$$

The determinants g and γ, formed respectively from the quantities g_{ik} and $\gamma_{\alpha\beta}$ are related to one another by

$$-g = g_{00}\gamma.$$ (see foot note*[1])

In some of the later applications it will be convenient to introduce the three-dimensional vector $\vec{g}$, whose covariant components are defined as

$$g_{\alpha} = -\frac{g_{0\alpha}}{g_{00}}.$$

Considering $\vec{g}$ as a vector in the space with metric (2) [i.e $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \dfrac{g_{0\alpha}g_{0\beta}}{g_{00}}$], we must define its contravariant components as

$$g^{\alpha} = \gamma^{\alpha\beta}g_{\beta} = -g^{0\alpha}.$$ (edit?)

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*[1] From $\gamma^{\alpha\beta} = -g^{ij}$ where I,j=0,1,..3 and $\alpha, \beta = 1,2,3,$ we get

$$\begin{bmatrix} \gamma^{11} & \gamma^{12} & \gamma^{13} \\ \gamma^{21} & \gamma^{22} & \gamma^{23} \\ \gamma^{31} & \gamma^{32} & \gamma^{33} \end{bmatrix} = -\begin{bmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ .. & .. & .. & .. \\ g^{30} & g^{31} & g^{32} & g^{33} \end{bmatrix}$$

Which gives on solving RHS

$$\begin{pmatrix} \gamma^{11} & \gamma^{12} & \gamma^{13} \\ \gamma^{21} & \gamma^{22} & \gamma^{23} \\ \gamma^{31} & \gamma^{32} & \gamma^{33} \end{pmatrix} = -g^{00}\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix}$$ which gives the results QED

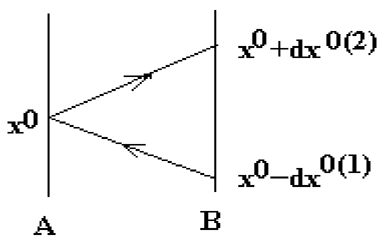
We also have the formula

$$g^{00} = -\frac{1}{g_{00}} g_\alpha g^\alpha \, .$$ (see foot note[2])

We now turn to **the definition of the concept of simultaneity[3] in the general theory of relativity**. **In other words, we discuss the question of the possibility of synchronizing clocks located at different points in space, i.e. the setting of a correspondence between the readings of such clocks.**

Such synchronization must obviously be achieved by means of an exchange of light signals between the two points.

We again consider the process of propagation of signals between two infinitely near points A and B. **We should regard as simultaneous with the moment x^0 at the point A that reading of the clock at point B which is half-way between the moments of departure and return of the signal to that point**, i.e. the



moment

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}\left(dx^{0(1)} + dx^{0(2)}\right).$$

[copied from previous papges

$$dx^{0(1)} = \frac{1}{g_{00}}\left[-g_{0\alpha}dx^\alpha - \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right]$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left[-g_{0\alpha}dx^\alpha + \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right]$$

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Substituting the corresponding expressions for $dx^{0(1)}$ and $dx^{0(2)}$ we find that the difference in the values of the time x^0 for two simultaneous events occurring at infinitely near points is given by

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$$g^\alpha g_\beta = -g^{0\alpha}\left(-\frac{g_{0\alpha}}{g_{00}}\right) = \frac{1}{g_{00}} = -g^{00}$$

3 This concept was first suggested by Galilio

$$\Delta x^0 = -\frac{g_{0\alpha} dx^\alpha}{g_{00}} = g_\alpha dx^\alpha .$$

This relation enables **to synchronize clocks in any infinitesimal region of space**. Carrying out a similar synchronization from the point A, we can synchronize clocks, i.e we can define simultaneity of events along an open curve.

However, synchronization of clocks along a closed contour turns out to be impossible in general. In fact, starting out along the contour and returning to the initial point we would obtain for Δx^0 a value different from zero. Thus it is impossible to synchronize clocks over all space. The exceptional cases are those reference systems in which all the components $g_{0\alpha}$ are equal to zero.

It should be emphasized that the impossibility of synchronization of all clocks is a property of the arbitrary reference system and not of the space-time itself. In any gravitational field, it is always possible (in infinitely many ways) to choose the reference system so that the three quantities $g_{0\alpha}$ become identically equal to zero, and thus make possible a complete synchronization of clocks.[Note by editir: because $g_{0\alpha}$ depends on the curvature of space coordinates and hence depend on transformation from one system to other.]

Even in the special theory of relativity, proper time elapses differently for clocks moving relative to each other. In the general theory of relativity, proper time elapses differently even at different points of space in the same reference system. **This means that the interval of proper time between two events occurring at the same point in space and the interval of time between two events simultaneous with these at another point in space are in general different from one another.**

7.2 The constant gravitational field

The motion of a free particle is determined in the special theory of relativity from **the principle of least action**,

$$\delta S = -mc\delta \int ds = 0 ,$$

[Note edit: In physics, the **principle of least action** – or, more accurately, the **principle of stationary action** – is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. The principle led to the development of the Lagrangian and Hamiltonian formulations of classical mechanics.]

according to which the particle moves so that its world line [4] is an extremal between a given pair of world points, in our case a straight line.

The motion of a particle in a gravitational field is determined by the principle of the least action in the same form, since the gravitational field is nothing but a change in the metric of the space time, manifesting itself only in a change in

[4] **Note edit:** In physics, the **world line** of an object is the unique path of that object as it travels through 4-dimensional spacetime.**]**

the expression for ds in terms of the dx^i. Thus, in a gravitational field the particle moves so that its **world point moves along an extremal**, or as it is called, a **geodesic line in the four space**; however, since in the presence of the gravitational field space-time is not Galilean, this line is not a straight line and the real spatial motion of the particle is neither uniform nor rectilinear.

In non-relativistic mechanics the motion of a particle in a gravitational field is determined by the Lagrangian

$$L = -m_0 c^2 + \frac{m_0 \mathrm{v}^2}{2} - m_0 \phi ,$$

where ϕ is the gravitational potential and we added the constant $-m_0 c^2$ so that in the absence of the gravitational field the Lagrangian $L = -m_0 c^2 + \frac{m_0 \mathrm{v}^2}{2}$ be exactly the same as that to which the corresponding relativistic function $L = -m_0 c^2 \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}$ reduces in the limit $\mathrm{v}/c \to 0$.

Consequently, the non-relativistic action function S for a particle in a gravitational field has the form

$$S = \int L dt = -m_0 c \int \left(c - \frac{\mathrm{v}^2}{2c} + \frac{\phi}{c} \right) dt .$$

Comparing this with the expression $S = -mc\int ds = 0$ we see that in the limiting case under consideration

$$ds = \left(c - \frac{\mathrm{v}^2}{2c} + \frac{\phi}{c} \right) dt .$$

Squaring and dropping terms which vanish for $c \to \infty$, we find

$$ds^2 = \left(1 + \frac{2\phi}{c^2} \right) c^2 dt^2 - d\vec{r}^{\,2} ,$$

where we have used the fact that $\vec{\mathrm{v}} dt = d\vec{r}$.

Thus in the limiting case of weak gravitational fields the component g_{00} of the metric tensor is

$$g_{00} = 1 + \frac{2\phi}{c^2} . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (1)$$

[NB: $d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0$, ds=c $d\tau$, dx^0=cdt where t is non-relativistic time]

A gravitational field is said to be constant if one can choose a system of reference in which all the components of the metric tensor are independent of the time coordinate x^0; the latter is then called the world time.

The choice of the world time is not completely unique. Thus, if we add to x^0 an arbitrary function of the space coordinates, the g_{ik} will still not contain x^0; this transformation corresponds to the arbitrariness in the choice of the time origin at each point in space. Of course the world time can be multiplied by an arbitrary constant, i.e. the units for measuring it are arbitrary.

Strictly speaking, **only the field produced by a single body can be constant**. In a system of several bodies, their mutual gravitational interaction will give rise to motion, as a result of which the field produced by them cannot be a constant.

If the body producing the field is fixed (in the reference system in which the g_{ik} do not depend on x^0), then both directions of time are equivalent. For a suitable choice of the time origin at all the points in space, the interval ds should not be changed when we change the sign of x^0. **Therefore all the components** $g_{0\alpha}$ **of the metric tensor must be identically equal to zero. Such constant gravitational fields are said to be static**.

However, for the field produced **by a body to be constant**, it is not necessary **for the body to be at rest**. Thus the field of an axially symmetric body rotating uniformly about its axis will also be constant. However, in this case the two time directions are no longer equivalent by any means-if the sign of the time is changed, the sign of the angular velocity is changed. Therefore in such constant gravitational fields (we shall call them **stationary fields**) the components $g_{0\alpha}$ of the metric tensor are in general different from zero.

The meaning of the **world time**[5] in a constant gravitational field is that an interval of world time between events at a certain point in space coincides with the interval of world time between any other two events at any other point in space, if these events are respectively simultaneous with the first pair of events. But to the same interval of world time x^0 there corresponds, at different points of space, different intervals of **proper time** τ.

The relation between world time and proper time can be written in the form

$$\tau = \frac{1}{c}\sqrt{g_{00}}x^0,$$

applicable to any finite time interval.

[5] Time measured by person moving

If the gravitational field is weak, we may use the approximate expression (1) to find

$$\tau = \frac{x^0}{c}\left(1 + \frac{\phi}{c^2}\right).$$

Thus **proper time elapses the more slowly the smaller the gravitational potential at a given point in space**, i.e., the larger its absolute value ($\phi < 0$). If one of two identical clocks is placed in a gravitational field for some time, the clock which has been in the field will thereafter appear to be slow.

In a static gravitational field the components $g_{0\alpha}$ of the metric tensor are zero. This means that in such a field **synchronization of clocks is possible over all space.** The element of the spatial distance in a static field is simply

$$dl^2 = -g_{\alpha\beta} dx^\alpha dx^\beta .$$

Let us consider the propagation of a light ray in a constant gravitational field. Let f be any quantity describing the field of the wave. For a plane monochromatic wave f has the form

$$f = a \exp\left[i\left(\vec{k} \cdot \vec{r} - \omega t + \alpha\right)\right],$$

where it is understood that we take the real part of the expression. Generally, the expression for the field can be written as

$$f = a \exp\left[i\psi\right].$$

In the case the wave is not plane, the amplitude a is a function of coordinates and time and the phase ψ, which is called **the eikonal**, does not have a simple form.

However, over small space regions and time intervals the eikonal ψ can be expanded in power series: to terms in first order we have

$$\psi = \psi_0 + \vec{r} \cdot \frac{\partial \psi}{\partial \vec{r}} + t \frac{\partial \psi}{\partial t},$$

(the origin for coordinates and time has been chosen within the space region and time interval under consideration; the derivatives are evaluated at the origin). Hence, by comparing this expression with the plane wave, we can write

$$\vec{k} = \frac{\partial \psi}{\partial \vec{r}} = \nabla \psi, \quad \omega = -\frac{\partial \psi}{\partial t}.$$

In the general relativistic case we can assume that the frequency of light is the derivative of the eikonal ψ with respect to the world time x^0/c,

$$\omega_0 = -c\frac{\partial\psi}{\partial x^0}.$$

Since the eikonal does not contain x^0 explicitly, the frequency ω_0 remains constant during the propagation of the light ray. The frequency measured in terms of the proper time is $\omega = -\frac{\partial\psi}{\partial\tau}$; **this frequency is different at different points in space**. From the relation

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{\partial x^0}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{c}{\sqrt{g_{00}}},$$

we have

$$\omega = \frac{\omega_0}{\sqrt{g_{00}}}.$$

In a weak gravitational field we obtain

$$\omega = \omega_0\left(1 - \frac{\phi}{c^2}\right).$$

We see that **the light frequency increases with increasing absolute value of the potential of the gravitational field, i.e. as we approach the bodies producing the field**; conversely, **as the light recedes from these bodies the frequency decreases.**

If a ray of light, emitted at a point where the gravitational potential is ϕ_1 has at that point the frequency ω, then upon arriving at a point where the potential is ϕ_2 will have a frequency (measured in units of the proper time at that point) equal to

$$\frac{\omega}{1-\frac{\phi_1}{c^2}}\left(1-\frac{\phi_2}{c^2}\right) = \omega\left(1+\frac{\phi_1-\phi_2}{c^2}\right)$$ ………………(foot note[6])

[6] $\omega_0 = \frac{\omega}{(1-\frac{\phi_1}{c^2})}$ *at potential* $\phi_1 \therefore$ *at potential* ϕ_2 equation is as above

A line spectrum emitted by some atoms located, for example, on the Sun, looks the same there as the spectrum emitted by the same atoms located on the Earth would appear on it.

If, however, **we observe on the Earth the spectrum emitted by the atoms located on the Sun**, then, as follows from what has been said above, **its lines appear to be shifted with respect to the lines of the same spectrum emitted on the Earth**.

Namely, each line with frequency ω will be shifted through the interval $\Delta\omega$ given by the formula

$$\Delta\omega = \frac{\phi_1 - \phi_2}{c^2}\omega, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{[imp]}$$

where ϕ_1 and ϕ_2 are the potentials of the gravitational field at the points of emission and observation of the spectrum, respectively.

If we observe on earth a spectrum emitted on the Sun or on the stars, then $|\phi_1| >> |\phi_2|$, and it follows that $\Delta\omega < 0$[7], i.e. the shift occurs in the direction of lower frequency.

This phenomenon is called gravitational red shift[3].

The occurrence of this phenomenon can be explained directly on the basis of the properties of the world time.

Because the field is constant, the interval of world time during which a certain vibration in the light wave propagates from one given point of space to another is independent of x^0.

Therefore it is clear that **the number of vibrations occurring in a unit interval of world time will be the same at all points along the ray.** But to one and the same interval of world time there correspond a larger and larger interval of proper time, the further away we are from the bodies producing the field. Consequently, **the frequency**, i.e. the number of vibrations per unit proper time, **will decrease as the light recedes from these masses.**

7.3 Energy and velocity

During the motion of a particle with rest mass m_0 in a constant gravitational field, its energy, defined as $-c\frac{\partial S}{\partial x^0}$, that is, the derivative of the action with respect to the world time, is conserved; this follows from the fact that in a constant field the

[7] Potential ϕ_1 is on sun and ϕ_2 is on earth

Hamiltonian does not depend on x^0. The energy defined in this way is the time component of the covariant four-vector of the momentum[4],

$$p_k = m_0 c u_k = m_0 c g_{kl} u^l \,.$$

In a static field

$$ds^2 = g_{00}\left(dx^0\right)^2 - dl^2 \,,$$

and we have for the energy of the particle

$$E = m_0 c^2 g_{00} \frac{dx^0}{ds} = m_0 c^2 g_{00} \frac{dx^0}{\sqrt{g_{00}\left(dx^0\right)^2 - dl^2}} \,.$$

We introduce the velocity

$$\mathrm{v} = \frac{dl}{d\tau} = \frac{cdl}{\sqrt{g_{00}}x^0} \,,$$

of the particle, measured in terms of the proper time[8] that is, by an observer located at the given point.

Then for the energy we obtain

$$E = \frac{m_0 c^2 \sqrt{g_{00}}}{\sqrt{1 - \dfrac{\mathrm{v}^2}{c^2}}} = mc^2 \sqrt{g_{00}} \,. \qquad (3)$$

This is the quantity which is conserved during the motion of a particle in a constant gravitational field.

In the limiting case of a weak gravitational field and low velocities, by substituting $g_{00} = 1 + \dfrac{2\phi}{c^2}$ in Eq. (3) we obtain approximately

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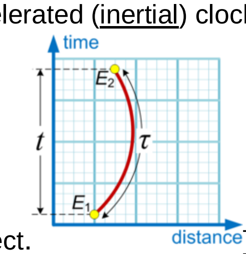
[8] In relativity, **proper time** is the elapsed time between two events as measured by a clock that passes through both events. The proper time depends not only on the events but also on the motion of the clock between the events. An accelerated clock will measure a smaller elapsed time between two events than that measured by a non-accelerated (inertial) clock between the same two events. The twin paradox is an example of this effect. 

],

$$E \approx m_0 c^2 + \frac{m_0 \mathrm{v}^2}{2} + m_0 \phi\,, \tag{3}$$

where $m_0\phi$ is the potential energy of the particle in the gravitational field.

The expression of energy given by Eq. (3) also remains valid for the case of a stationary field.

If the particle departs from point A at the moment of world time x^0 and arrives at the infinitesimally distant point B at the moment $x^0 + dx^0$, then to determine the velocity we must take not the time interval $\left(x^0 + dx^0\right) - x^0 = dx^0$, but rather the difference between $x^0 + dx^0$ and the moment $x^0 - \left(g_{0\alpha} / g_{00}\right)dx^\alpha$, which is simultaneous at the point B with the moment x^0 at the point A:

$$\left(x^0 + dx^0\right) - \left(x^0 - \frac{g_{0\alpha}}{g_{00}}dx^\alpha\right) = dx^0 + \frac{g_{0\alpha}}{g_{00}}dx^\alpha\,.$$

Multiplying by $\sqrt{g_{00}}/c$, we obtain the corresponding interval of proper time, so that the velocity is

$$\mathrm{v}^\alpha = \frac{c\,dx^\alpha}{\sqrt{h}\left(dx^0 - g_\alpha dx^\alpha\right)},$$

where we have introduced the notations

$$h = g_{00},\ g_\alpha = -\frac{g_{0\alpha}}{g_{00}}\,.$$

The covariant components of the velocity $\vec{\mathrm{v}}$ form a three-dimensional vector in the space with metric $\gamma_{\alpha\beta}$ and correspondingly the square of this vector is

$$\mathrm{v}_\alpha = \gamma_{\alpha\beta}\mathrm{v}^\beta\,,\ \mathrm{v}^2 = \mathrm{v}_\alpha \mathrm{v}^\alpha\,.$$

With such a definition, the interval ds is expressed in terms of the velocity in the usual fashion:

$$\begin{aligned} ds^2 &= g_{00}\left(dx^0\right)^2 + 2g_{0\alpha}dx^0 dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta = h\left(dx^0 - g_\alpha dx^\alpha\right)^2 - dl^2 \\ &= h\left(dx^0 - g_\alpha dx^\alpha\right)^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right). \end{aligned}$$

The components of the four-velocity

$$u^i = \frac{dx^i}{ds},$$

are

$$u^0 = \frac{1}{\sqrt{h}\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} + \frac{g_\alpha \mathrm{v}^\alpha}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}},\ u^\alpha = \frac{\mathrm{v}^\alpha}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}.$$

The energy is

$$E = m_0 c^2 g_{0i} u^i = m_0 c^2 h\left(u^0 - g_\alpha u^\alpha\right).$$

After substituting the values of the four-velocity we obtain again the expression (3).

Notes

[1] For a discussion of the relation between the field and geometrical aspects of gravity see V. A. Fock, The theory of space, time and gravitation, New York, Pergamon Press, 1959 and R. P. Feynman, Lectures on gravitation, Pasadena, California Institute of Technology, 1971.

[2] The definitions of the proper time and space intervals follows L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971. A different approach is presented in S. Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity New York, Wiley, 1972.

[3] A more extended analysis of the gravitational redshift effect, with emphasis on the observational data, is given in H. C. Ohanian, Gravitation and spacetime, W.W. Norton and Comp., New York, 1976.

[4] The definitions of the energy and momentum in a gravitational field follow L. D. Landau and E. M. Lifshitz, The Classical theory of fields, Oxford, Pergamon Press, 1971.