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A NOTE ON THE PROPAGATION OF SHOCK WAVES IN AN EXPONENTIALLY DECREASING MEDIUM

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Introduction : Problem of propagation of shock waves in non-uniform medium is of great interest for exploring the effect of explosions in stars and atmosphere of the earth. Earlier many authors have considered this problem by taking different density variations, by the methods of similarity solutions. Sakurai [1] has discussed the problem of propagation of weakening shocks in stars by expanding all the thermodynamical parameters in the form of a series in terms of powers of $(R_9/\overline{R})$ where R_0 is the distance of the shock front from the point of explosion and $\overline{R}$ is the distance depending on the strength of the explosion. Carrus et. al. [2], Kopal [3] and Bhatnagar et. al. [4] have transformed nonlinear equations of motion into linear ones by introducing a similarity variable $\xi=r^{\phi}\, t^{\psi}$ and have found the variation of self-similar fluid velocity, density and pressure behind the shock front, with the help of numerical methods.

Propagation of shock waves in an exponentially decreasing medium has also been studied by various authors such as, Grover and Hardy [5], Kompaneets [6], Andriankin et. al. [7], Raizer [8] and Hayes [9. 10].

Bhatnagar and Sachdev [11] applied Whitham's Rule [12] to the propagation of shock waves in an isothermal, self-gravitating, radiating gas sphere. A differential equation for the variation of Mach number and shock distance was found. This equation was integrated numerically for different models of the stars.

In this paper by using Whitham's Rule [12] an attempt has been made to find an analytical approximate relation for the Mach number in terms of distance of the shock front, in a one dimensional plane model in which density and temperature is varying exponentially. Density and temperature in the medium are varying as $\rho_0=\rho_c\, e^{-\alpha x}$, $T=T_c e^{-\beta x}$, where α and β are some constants having dimensions of inverse of a distance, and x is the distance measured from the point of explosion. Variations of the shock velocity and the Mach number is obtained in terms of the shock distance, for $\lambda=0, \gamma=1$ and $\lambda=1, \gamma=4/3, 7/5$ where $\lambda=\beta/\alpha$. So that $\lambda=0$ is an isothermal case and $\lambda=1$, a case with decreasing temperature. In all the three cases, it is shown that the shock velocity and the Mach number increases as the shock propagates from the point of explosion. Graph showing the variation of the shock velocity U and distance R is drawn for $\gamma=1$, 4/3 and 7/5 where U and R are dimensionless. An approximate relation for the Mach number and the shock velocity is found in terms of the distance of the shock front.

Formulation of the Problem : Let us assume a medium in which thermodynamical parameters are varying in parallel planes from a fixed plane denoted by $x=0$, according to the law,

$$\rho_0=\rho_c\, e^{-\alpha x},\ T_0=T_c\, e^{-\beta x}, \qquad \ldots(1)$$

where ρ_c, T_0 are density and absolute temperature at a distance x measured normal to the planes of variation and ρ_c, T_c their values at the plane $x=0$. α, β are some constants having dimensions of inverse of a distance. We assume that at time $t_0=0$, there is an explosion at point O in $x=0$ plane. Due to the explosion, a shock wave is created which moves in all the directions. We consider only that portion which moves normal to the plane of variation of density and temperature. For simplicity shock front is assumed to be plane. Let U_0, M_0 be the shock velocity and the Mach number of the shock when it reaches at a distance R_0, at time t_0. We take medium to be such that it holds gas law,

$$p_0=R\rho_0T_0$$

where R is the gas constant and p_0 is the pressure. From equation (1) we have by using gas law,

$$p_0(x)=p_c\, e^{-(\alpha+\beta)x} \qquad \ldots(2)$$

where $p_c=R\rho_cT_c$

If u_{02}, p_{02}, ρ_{02} denote fluid velocity, pressure and density behind the shock front, jump conditions across the shock front are,

$$u_{02}\,(R_0,\, t_0)=\frac{2c_0}{\gamma+1}\,\{M_0-M_0^{-1}\}$$

$$p_{02}\,(R_0,\, t_0)=\frac{\gamma p_0}{(\gamma+1)}\left\{2M_0^2-\frac{\gamma-1}{\gamma}\right\} \qquad \ldots(3)$$

$$\rho_{02}\,(R_0,\, t_0)=\frac{(\gamma+1)\,\rho_0M_0^2}{\{2+(\gamma-1)\,M_0^2\}}$$

where $M_0=\dfrac{U_0}{c_0}$ and $c_0=\sqrt{\gamma p_0/\rho_0}$

and R_0 is the distance of the shock front from point O.

We define dimensionless parameters p, ρ, U, c, p_2, ρ_2, u_2, c_2, as

$$p=p_0/p_c,\ \rho=\rho_0/\rho_c,\ U=U_0/c_c,\ c=c_0/c_c$$

$$p_2=p_{02}/p_c,\ \rho_2=\rho_{02}/\rho_c,\ u_2=u_{02},\ c_2=c_{02}/c_c \qquad \ldots(4)$$

where $c_c=(p_c/\rho_c)^{1/2}$

and dimensionless distances r, R and time t as,

$$r=\alpha x,\ R=\alpha r_0,\ t=\alpha c_c\, t_0. \qquad \ldots(5)$$

Substituting the relations (1), (4) and (5) in relation (3) we get,

$$u_2\,(R,\, t)=\frac{2c}{\gamma+1}\,\{M-M^{-1}\}$$

$$p_2\,(R.\, t)=\frac{\gamma p}{(\gamma+1)}\,f\,(M) \qquad \ldots(6)$$

$$\rho_2\,(R,\, t)=\frac{(\gamma+1)\,\rho M^2}{g\,(M)}$$

where

$$M=U/c=U_0/c_0=M_0$$

$$f\,(M)=\left\{2M^2-\frac{\gamma-1}{\gamma}\right\} \qquad \ldots(7)$$

$$g(M)=\{2+(\gamma-1)\ M^2\}$$

R is the dimensionless distance of shock front from point O. Relations (6) are Rankine-Hugoniot relations in dimensionless variables.

The Discussion of the Problem : We shall now apply Whitham's Rule to the relations (6). The equations of motion along the positive characteristic axis $\frac{dR}{dt}=u_2+c_2$ is given as,

$$dp_2+\rho_2 c_2\ du_2=0. \qquad \ldots(8)$$

Now since by Whitham's Rule, variations of parameters behind the shock front are parallel to the positive characteristic axis behind the shock front, we put relations (6) in relation (8) and get,

$$\{f(M)+(M^2-1)\ h(M)\}\ \frac{dp}{p}-(M^2-1)\ h(M)\frac{d\rho}{\rho}$$
$$+\{4h^2(M)+2(M^2+1)\ h(M)\}\ \frac{dM}{M}=0, \qquad \ldots(9)$$

where
$$h(M)=\sqrt{\frac{\gamma f(M)}{g(M)}}. \qquad \ldots(10)$$

By making use of the relations

$$\rho=e^{-R},\ p=e^{-(\lambda+1)R} \qquad \ldots(11)$$

at $r=R$, where $\lambda=\frac{\beta}{\alpha}$ into the relations (9), we get a differential relation in M and R as,

$$2\ \frac{dM}{dR}=MK(M), \qquad \ldots(12)$$

where

$$K(M)=\frac{(1+\lambda)\ f(M)+\lambda\ (M^2-1)\ h(M)}{\{2M^2+(M^2+1)\ h(M)\}}. \qquad \ldots(13)$$

In the table, we have given the values of $K(M)$ for $\lambda=0$, $\gamma=1$ and $\lambda=1$, $\gamma=4/3$, $7/5$, as M varies from 1 to infinity. It is found that variation in $K(M)$ as $M\geqslant 2$ is small for $\lambda=1$, $\gamma=4/3$, $7/5$ but for $\lambda=0$, $\gamma=1$, $K(M)\to 0$ as $M\to\infty$ and variation is not so small. Hence in former case, we take variations in $K(M)$ negligible as M varies. We consider two cases, first a general case i.e., $\lambda=1$, $\gamma=4/3$, $7/5$ and second an isothermal case, *i.e.*, $\lambda=0$, $\gamma=1$. We also assume that value of Mach number at point O is two.

General Case : If we integrate equation (12), and assume $K(M)$ to be constant for the purpose of integration, we get,

$$R=\frac{2}{K(M)}\ l_n\left(\frac{M}{M_c}\right) \qquad \ldots(14)$$

where M_c is the value M at $R=0$. Relation (14) can also be written as,

$$M=M_c\ e^{K(M)R/2} \qquad \ldots(15)$$

also from (15) we have an expression for shock velocity as

$$U=\sqrt{\gamma}M_c\ e^{-(\lambda-K)R/2}. \qquad \ldots(16)$$

It is to be noted that (15) and (16) hold only for $\lambda=1$, since variation in $K(M)$ is not small for $\lambda=0$.

Isothermal Case : Since variations in $K(M)$ is not small for $\lambda=0$, $\gamma=1$ and also $K(M)\to 0$ as $M\to\infty$, we cannot apply above method for this case. But it is easily seen that $K(M)$ is very much simplified for $\lambda=0$. Expression for $K(M)$ for $\lambda=0$ and $\gamma=1$ is

$$K(M)=\frac{2M}{(M+1)^2}. \qquad \ldots(17)$$

Therefore equation (12) becomes

$$\frac{dM}{dR}=\left(\frac{M}{M+1}\right)^2. \qquad \ldots(18)$$

Integrating (16) and using the condition, when $R=0$, $M=M_c$, we get

$$M^2 e^{(M^2-1)/M}=Ae^R, \qquad \ldots(19)$$

where $$A=M_c^2 e^{(M_c^2-1)/M_c}.$$

Also, expression for sound velocity is given by

$$C=\sqrt{\gamma}e^{-\lambda R/2} \qquad \ldots(21)$$

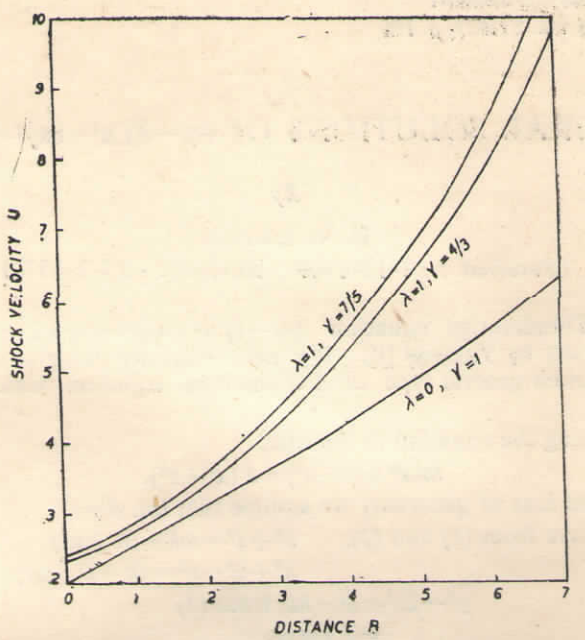
and is unity for isothermal case. Relations (19) and (21) combined give the shock velocity.

TABLE

VARIATION OF K(M) WITH M.

M \ K(M)		$\lambda=0$	$\lambda=1$	
		$\gamma=1$	$\gamma=7/5$	$\gamma=4/3$
1		·500	·857	·875
2		·444	1·238	1·237
3		·375	1·338	1·331
4		·320	1·372	1·372
5		·278	1·396	1·383
6		·245	1·406	1·393
7		·219	1·409	1·398
8		·197	1·417	1·402
9		·180	1·420	1·405
10	·165	·165	1·422	1·406
15		·117	1·427	1·411
20		·090	1·429	1·413
25		·074	1·429	1·413
50		·038	1·432	1·414
		·000	1·432	1·414

In the figure we have drawn the shock velocity U against the distance R, for $\lambda=1, \gamma=4/3$, 7/5 ahd $\lambda=0, \gamma=1$. It is shown that shock velocity increases slowly for isothermal case *i.e.*, for $\lambda=0$, but increases sharply for $\lambda=1$. But in both the cases the strength of the shock wave increases as it moves in an exponentially decreasing medium. By Whitham's method we have been able to show that shock velocity and the Mach number of the shock velocity increases as it propagates in the medium with decreasing density. While



by the method of similarity solution this was not possible and expression for Mach number with that method shows that it is constant. It is our intention to extend this work to the atmosphere of earth elsewhere.

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