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A NOTE ON THE PROPAGATION OF SHOCK WAVES IN NON-HOMOGENEOUS MEDIUM

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Introduction : Sakurai [3], Carrus, et. al. [1], Kopal [2], and many other authors have discussed the problem of propagation of shock waves in spherical non-homogeneous model by using self-similarity methods. Sakurai has taken density as solution of Lane Emden equation, while other authors have taken density varying as power of inverse distance, with centre of the star as origin. Kopal and Carrus et. al. have introduced a new variable $\xi = r^{-\lambda} t$ where $\lambda = \alpha/2$ and thus have transformed non-linear equations of motion into linear one. By taking ξ=constant at the shock front they have found Mach number of the shock which comes out to be constant.

In the present paper the problem of propagation of shock waves in one dimensional plane model in which density is varying as, $\rho_0 = \bar{\rho} r^{-\alpha}$ is considered, by using Whitham's Rule [4]. Results of Kopal, Carrus and other's are derived directly, with little modifications. It is shown that Mach number and shock velocity increases as the shock propagates into the decreasing density medium. Analytic relations are obtained for Mach number and shock velocity.

Basic Equations : We consider a plane one dimensional medium, acted upon by a force $F_0(x)$ given by $F_0(x) = F_c\, x^{(1-\alpha)}$, F_c being value of $F_0(x)$ at point $x=0$. We denote this point by 0. It is seen from the expression for $F_0(x)$ that point 0 is a point of discontinuity and $F_0(x)$ becomes infinity at this point. To avoid discontinuity at 0, we assume that there is a small region of length x_* from 0, in which density of medium is constant and is of value ρ_c. Due to force $F_0(x)$, density is also varying in the medium. Let its form is given by $\rho_0 = \rho_c x^{-\alpha}$ where ρ_c is the density at point 0. Force $F_0(x)$ and density $\rho_0(x)$ are varying from point 0 in parallel planes.

If u_0, p_0, ρ_0, t_0 are fluid velocity, pressure, density and time at a distance x from point 0, then equations of motion are,

$$\frac{\partial \rho_0}{\partial t_0} + u_0 \frac{\partial \rho_0}{\partial x} + \rho_0 \frac{\partial u_0}{\partial x} = 0 \qquad \ldots(1)$$

$$\frac{\partial u_0}{\partial t_0} + u_0 \frac{\partial u_0}{\partial x} + \frac{1}{\rho_0} \frac{\partial p_0}{\partial x} + F_0(x) = 0 \qquad \ldots(2)$$

and conservation of entropy gives

$$\frac{\partial}{\partial t_0}(p_0 \rho_0^{-\gamma}) + u_0 \frac{\partial}{\partial x}(p_0\, \rho_0^{-\gamma}) = 0. \qquad \ldots(3)$$

Here we have assumed that entropy is constant along the stream lines.

We introduce a distance $\bar{R}$, such that at $x = \bar{R}$, force, pressure and density become $\bar{F}$, $\bar{p}$ and $\bar{\rho}$. Then since $F_0(x) = F_c x^{(1-\alpha)}$, we have

$$F_c = \bar{F}\, \bar{R}^{(\alpha-1)} \qquad \ldots(4)$$

and

$$\rho_c = \bar{\rho}\, \bar{R}^{\alpha}. \qquad \ldots(5)$$

We introduce dimensionless variables given by,

$$F=F_0/\bar{F},\ p=p_0/2(\alpha-1)\ \bar{p},\ \rho=\rho_0/\bar{\rho},\ u=\frac{u_0}{\bar{c}}$$

$$r=x/\bar{R} \text{ and } t=(\bar{c}/\bar{R})t_0 \qquad \ldots(6)$$

where

$$\bar{c}^2=\frac{2(\alpha-1)\bar{p}}{\bar{\rho}} \text{ and } \bar{F}=\frac{2(\alpha-1)\bar{p}}{\bar{\rho}\,\bar{R}}.$$

Then equations (1)–(3) become

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\frac{\partial u}{\partial r}=0. \qquad \ldots(7)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}+F(r)=0. \qquad \ldots(8)$$

$$\frac{\partial}{\partial t}(p\rho^{-r})+u\frac{\partial}{\partial r}(p\rho^{-r})=0. \qquad \ldots(9)$$

Force and density in equilibrium state become,

$$F=r^{(1-\alpha)},\ \rho=r^{-\alpha}. \qquad \ldots(10)$$

From (8) we get in equilibrium,

$$\frac{dp}{dr}=-\rho F.$$

Integrating this we get,

$$p=\frac{r^{2-2\alpha}}{2(\alpha-1)}. \qquad \ldots(10a)$$

In order pressure to be positive, we must have $\alpha>1$.

We assume that there is an explosion at point 0 and a shock propagates along a line which is normal to the plane of variation of density and pressure. When shock reaches a distance r_*, where $r_*=x_*/\bar{R}$, at time t_*, let its strength be given by $M=2$. Thus we eliminate the mathematical discontinuity at point $r=0$.

Let quantities behind the shock front be denoted by subscript 2. As shock crosses the point where u, p, f are the values of velocity (dimensionless), pressure and density, jumps in these quantities across the shock at point $r=R$ is,

$$u_2(R,t)=\frac{2c}{(r+1)}(M-M^{-1}) \qquad \ldots(11)$$

$$p_2(R,t)=\frac{rp}{(r+1)}f(M) \qquad \ldots(12)$$

$$\rho_2(R,t)=\frac{(r+1)\rho M^2}{g(M)} \qquad \ldots(13)$$

where $M=U/c$, U being shock velocity and R being shock distance from point 0. And

$$f(M)=\left\{2M^2-\frac{r-1}{r}\right\} \qquad \ldots(14)$$

$$g(M)=\{2+(r-1)M^2\} \qquad \ldots(15)$$

$$c^2=rp/\rho. \qquad \ldots(16)$$

Here we have taken fluid to be stationary in front of the shock surface. Relations (11) to (13) are known as Rankine-Hugoniot relations.

Discussion of the Problem: Now we shall apply Whitham's Rule [4] to get a relation for shock velocity. Equations of motion along the positive characteristic axis $\frac{dR}{dt}=u_2+c_2$ is,

$$dp_2+\rho_2 c_2\, du_2+\frac{\rho_2 c_2\, F(R)dR}{u_2+c_2}=0. \qquad \ldots(17)$$

Substituting values of P_2, ρ_2, u_2 from (11) to (13) in relation (17), we get after some simplifications,

$$2\,\{2M^2+(M^2+1)\,h(M)\}\,\frac{dM}{M}+\{f(M)+(M^2-1)\,h(M)\}\,\frac{dp}{p}$$
$$-(M^2-1)\,h(M)\frac{d\rho}{\rho}+\frac{(r+1)^2\,M^2\,h(M)\,F(R)\,dR}{c^2\,\{2(M^2-1)+\sqrt{rfg}\}}=0 \qquad \ldots(18)$$

where $$h(M)=\left\{\frac{rf(M)}{g(M)}\right\}^{1/2}. \qquad \ldots(19)$$

Using values of p and f from (10a) and (10) in (18) we get,

$$2R\,\frac{\partial M}{\partial R}+\frac{2R^{\alpha/2}}{\mu M}\,\frac{\partial M}{\partial t}-MK(M)=0. \qquad \ldots(20)$$

μ being a constant given by $\sqrt{\frac{r}{2(\alpha-1)}}$ and,

$$K(M)\frac{\left[2(\alpha-1)\left\{f(M)-\frac{M^2\,h(M)\,\frac{(r+1)^2}{r}}{\{2(M^2-1)+\sqrt{rfg}\}}\right\}+(\alpha-2)\,(M^2-1)\,h(M)\right]}{\{2M^2+(M^2+1)\,h(M)\}}. \qquad \ldots(21)$$

In table, I, values of $K(M)$ are computed for $\alpha=1\cdot5$, 2, $2\cdot5$ and $r=1\cdot4$, 1·5, 1·667 and it is found that the variation in $K(M)$ is small as M varies two to infinity. As Mach number increases from one to two, there is an abrupt change in the value of $K(M)$, which is due to the presence of the force $F\,(R)$. Since for small values of M shock is weak and hence force F is dominating. At $M=1$, velocity behind the disturbance front is zero, since it is zero in front of it. Hence force balances the pressure term and $K(M)$ is zero. But as shock becomes stronger and stronger, velocity behind the shock also increases and so does the pressure. Moreover force is decreasing

TABLE I

	$\alpha=1\cdot5$			$\alpha=2$			$\alpha=2\cdot5$		
M	$\gamma=1\cdot4$	$\gamma=1\cdot5$	$\gamma=5/3$	$\gamma=1\cdot4$	$\gamma=1\cdot5$	$\gamma=5/3$	$\gamma=1\cdot4$	$\gamma=1\cdot5$	$\alpha=5/3$
1	0	0	0	0	0	0	0	0	0
1·5	0·144	0·163	0·156	0·519	0·521	0·505	0·828	0·900	0·855
2	0·167	0·177	0·189	0·646	0·662	0·683	1·125	1·148	0·992
3	0·166	0·183	0·204	0·764	0·789	0·821	1·362	1·395	1·438
4	0·160	0·181	0·207	0·815	0·837	0·973	1·441	1·491	1·539
5	0·156	0·179	0·208	0·826	0·858	0·828	1·496	1·537	1·595
10	0·154	0·176	0·208	0·863	0·869	0·933	1·572	1·602	1·657
	0·146	0·174	0·208	0·866	0·899	0·944	1·576	1·624	1·687

Variation of $K(M)$ versus M.

as R increases. Hence pressure term and inertia term dominate. At very large distance from the centre, effect of force $F(R)$ is negligible and also variations in $K(M)$ become negligible. Due to these small variations in $K(M)$, if we take $K(M)$ to be constant during the process of integration, it simplifies the results to a great extent. Integrating partial differential equation (20) as R, t vary from r_*, t_* to current value R, t we get,

$$M(R,t)=\frac{R^{\alpha/2}}{\left\{r_*^{\alpha/2}+\mu\frac{\alpha-K}{2}(t-t_*)\right\}} \qquad \ldots(22)$$

also we have

$$\frac{dR}{dt}=Mc. \qquad \ldots(23)$$

Putting values of M from (22) and c from (10), (10a) in (23) we get

$$\frac{dR}{dt}=\frac{\mu R}{\{r_*^{\alpha/2}+\mu\lambda(t-t_*)\}} \qquad \ldots(24)$$

where

$$\lambda=\frac{\alpha-K}{2}. \qquad \ldots(25)$$

Relation (24) expresses shock velocity in terms of R and t both. Eliminating t from (22) and (24) we get ...(26)

$$M=(R/r_*)^{K/2}$$

and

$$U=\mu R^{(1-\lambda)}/r_*^{K/2}. \qquad \ldots(27)$$

Relations (26) and (27) give Mach number and shock velocity as a function of R and r_*.

Results (26) and (27) are of course true for $M\geqslant 2$, since variation of $K(M)$ is not small for $M<2$. These relations show that M increases with R. This means shock becomes stronger and stronger as it propagates in decreasing density medium. Results (27) and (26) are remarkably true for higher values of M.

Comparison of Results with the Previous Work : The results (22), (24), (26) and (27) are derived on the assmuptions that the shock starts at a distance r_* from the point 0, which is a point of discontinuity, and has not been considered in the problem. However in some results if we take an approximation that r_* and t_* go to zero in the limit, we get the results which have already been derived by the earlier authors. If we put r_* and t_* equal to zero in (22), we get

$$M(R,t)=R^{\alpha/2}/u\lambda t \qquad \ldots(28)$$

and from (24) we get

$$\frac{dR}{dt}=R/\lambda t. \qquad \ldots(29)$$

Expression (29) is same as derived by Kopal [2], except that $\lambda=\frac{\alpha}{2}$ in his case.

Now if we integrate (29) from r_*, t_* to R, t we get,

$$R^{-\lambda}t=r_*^{-\lambda}t_*=\eta_* \qquad \ldots(30)$$

where η_* is a constant. Using (30) in (26) and (29) we get,

$$M(R,t)=R^{K/2}/\mu\lambda\eta_* \qquad \ldots(31)$$

and

$$U(R, t)=\frac{R^{(1-\lambda)}}{\lambda\eta_*} \qquad \ldots(32)$$

Result (31) is parallel to the one derived by Kopal, except for the factor $R^{K/2}$, which is only responsible for the variation of M.

It is concluded from the above discussion that if we take $K=0$, results tally with those of earlier authors. Although the results derived above are sufficiently approximate, since we have taken K to be constant approximately, yet they lead us to more physical conclusions that shock becomes stronger and stronger as it propagates outward and is not constant as shown in the earlier works. Beauty of this technique, is that we get analytical relations for Mach number and shock velocity in terms of distance of the shock front from the point of explosion. Also by using this technique we avoid similarity solution method. The same technique can be generalised for the spherical case.

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