Module 6

Knowledge Representation and Logic – (First Order Logic)

Lesson 14

First Order Logic - II

6.2.5 Herbrand Universe

It is a good exercise to determine for given formulae if they are satisfied/valid on specific L-structures, and to determine, if they exist, models for them. A good starting point in this task, and useful for a number of other reasons, is the **Herbrand Universe** for this set of formulae. Say that {F01 .. F0n} are the individual constants in the formulae [if there are no such constants, then introduce one, say, F0]. Say that {F1 .. Fm} are all the non 0-ary function symbols occurring in the formulae. Then the set of (constant) terms obtained starting from the individual constants using the non 0-ary functions, is called the Herbrand Universe for these formulae.

For example, given the formula (P x A) OR (Q y), its Herbrand Universe is just {A}. Given the formulae (P x (F y)) OR (Q A), its Herbrand Universe is {A (F A) (F (F A)) (F (F (F A))) ...}.

Reduction to Clausal Form

In the following we give an algorithm for deriving from a formula an equivalent clausal form through a series of truth preserving transformations.

We can state an (unproven by us) theorem:

Theorem: Every formula is equivalent to a clausal form

We can thus, when we want, restrict our attention only to such forms.

6.2.6 Deduction

An **Inference Rule** is a rule for obtaining a new formula [the **consequence**] from a set of given formulae [the **premises**].

A most famous inference rule is **Modus Ponens**:

```
                {A, NOT A OR B}
               ---------------
                      B
For example:
               {Sam is tall,
                if Sam is tall then Sam is unhappy}
               ------------------------------------
                 Sam is unhappy
```

When we introduce inference rules we want them to be **Sound**, that is, we want the consequence of the rule to be a logical consequence of the premises of the rule. Modus Ponens is sound. But the following rule, called **Abduction** , is not:

```
               {B, NOT A OR B}
               --------------
                     A
```

is not. For example:

```
John is wet

If it is raining then John is wet
---------------------------------
It is raining
```

gives us a conclusion that is usually, but not always true [John takes a shower even when it is not raining].

A **Logic** or **Deductive System** is a language, plus a set of inference rules, plus a set of logical axioms [formulae that are valid].

A **Deduction or Proof or Derivation** in a deductive system D, given a set of formulae GAMMA, is a a sequence of formulae B1 B2 .. Bn such that:

- for all i from 1 to n, Bi is either a logical axiom of D, or an element of GAMMA, or is obtained from a subset of {B1 B2 .. Bi-1} by using an inference rule of D.

In this case we say that Bn is **Derived** from GAMMA in D, and in the case that GAMMA is empty, we say that Bn is a **Theorem** of D.

6.2.7 Soundness, Completeness, Consistency, Satisfiability

A Logic D is **Sound** iff for all sets of formulae GAMMA and any formula A:

- if A is derived from GAMMA in D, then A is a logical consequence of GAMMA

A Logic D is **Complete** iff for all sets of formulae GAMMA and any formula A:

- If A is a logical consequence of GAMMA, then A can be derived from GAMMA in D.

A Logic D is **Refutation Complete** iff for all sets of formulae GAMMA and any formula A:

- If A is a logical consequence of GAMMA, then the union of GAMMA and (NON A) is inconsistent

Note that if a Logic is Refutation Complete then we can enumerate all the logical consequences of GAMMA and, for any formula A, we can reduce the question if A is or not a logical consequence of GAMMA to the question: the union of GAMMA and NOT A is or not consistent.

We will work with logics that are both Sound and Complete, or at least Sound and Refutation Complete.

A **Theory** T consists of a logic and of a set of Non-logical axioms. For convenience, we may refer, when not ambiguous, to the logic of T, or the non-logical axioms of T, just as T.

The common situation is that we have in mind a well defined "world" or set of worlds. For example we may know about the natural numbers and the arithmetic operations and relations. Or we may think of the block world. We choose a language to talk about these worlds. We introduce function and predicate symbols as it is appropriate. We then introduce formulae, called **Non-Logical Axioms**, to characterize the things that are true in the worlds of interest to us. We choose a logic, hopefully sound and (refutation) complete, to derive new facts about the worlds from the non-logical axioms.

A **Theorem** in a theory T is a formula A that can be derived in the logic of T from the non-logical axioms of T.

A Theory T is **Consistent** iff there is no formula A such that both A and NOT A are theorems of T; it is **Inconsistent** otherwise.
If a theory T is inconsistent, then, for essentially any logic, any formula is a theorem of T. [Since T is inconsistent, there is a formula A such that both A and NOT A are theorems of T. It is hard to imagine a logic where from A and (NOT A) we cannot infer FALSE, and from FALSE we cannot infer any formula. We will say that a logic that is at least this powerful is **Adeguate**.]

A Theory T is **Unsatisfiable** if there is no structure where all the non-logical axioms of T are valid. Otherwise it is **Satisfiable**.

Given a Theory T, a formula A is a **Logical Consequence of T** if it is a logical consequence of the non logical axioms of T.

Theorem: If the logic we are using is sound then:

1. If a theory T is satisfiable then T is consistent
2. If the logic used is also adequate then if T is consistent then T is satisfiable
3. If a theory T is satisfiable and by adding to T the non-logical axiom (NOT A) we get a theory that is not satisfiable Then A is a logical consequence of T.
4. If a theory T is satisfiable and by adding the formula (NOT A) to T we get a theory that is inconsistent, then A is a logical consequence of T.