Chapter 1

Review of Basic Principles

1.1 Overview

This chapter is concerned with the basic principles of classical mechanics. The first section begins with a reminder of Newton's Laws and continues with an examination of some even more fundamental postulates and the principles of inertia and relativity, which may not be so familiar. These ideas seem so simple and straightforward to us today that it is difficult to appreciate the enormous struggle required to conceive of, refine, and experimentally verify them. Most systems of interest in mechanics contain more than a single particle, of course. Section 1.4 shows that systems of particles (and therefore extended bodies) behave in many respects as single particles. The conservation principles of momentum, angular momentum, and energy make their first, but far from last, appearance. Not all dynamics, at least on the surface, is conservative. Section 1.7 briefly deals with the problem of dissipative forces and approximate ways to handle them. The gravitational force, an example of a conservative force, is discussed in section refgravity. We show that a spherically symmetric body may be treated as though it were concentrated at its center as far as its gravitational influence is concerned. This is closely related to the Poisson equation.

There are few examples in this chapter, which is largely due to the review character[1] The Notes at the end give cross references to Marion and Thornton, 4th ed. (hereafter known as MT), where more can be found.

1.2 Foundations

1.2.1 Newton's Laws

Tradition demands that we begin with the hallowed

[1] I'm trying to think of some other excuses.

Newton's Laws of Motion

1. A particle free of external forces moves at constant velocity.
2. Under the influence of an external force $\mathbf{F}$, the motion of a particle is such that the time rate of change of its momentum $\mathbf{p} = m\mathbf{v}$ is given by the force: $d\mathbf{p}/dt = \mathbf{F}$.
3. The force exerted by one body upon a second is minus the force exerted by the second upon the first. I.e., labelling the bodies a and b, $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$.

You are probably able to recite these in approximately this form even half asleep. They are very terse statements, and there is, unspoken, a great deal woven around and through them. For instance, the concepts of *mass* and *force* appear here in an undefined form. Did we forget to define them, or are they defined *by* Newton's Laws? We have some rough intuitive notion of what these things are, but it is difficult not to reach the conclusion that they are to some extent defined by the Laws of Motion.

1.2.2 Reference Frames

There are some assumptions about space and time which underlie the entire structure of classical mechanics. These are often left unstated because they are mostly in accord with our everyday intuition. But there is value in examining them since we now know that that they are not fundamentally correct, at least twice over.

Assumptions about space and time

- Space is three-dimensional and Euclidean.
- Time is one-dimensional.
- Different observers will always agree upon:
  - The time interval between a pair of events, and
  - The distance between two simultaneous events.

The last item really means complete agreement on the Euclidean geometry associated with simultaneous events. This is the most economical way of putting it since agreement on legths implies agreement on angles (the lengths of the three sides of a triangle determine all the angles). But why the 'simultaneous'? Put it this way: what's the distance between Woolsthorpe England on December 25 (Newton's birthday[2]) and Brooklyn New York on July 12 (Bill Cosby's birthday)? A few thousand miles? How about a couple hundred million? To an astronomer, the latter answer might be more natural. You see the problem. This is where the idea of a **reference frame** comes in. A reference frame tells us how to

[2]At the time, England was still using the old Julian calendar. Much of Europe had already adopted the Gregorian calendar, according to which Newton was born sometime in January.

sew together the Euclidean 3-spaces corresponding to different times.

The points of three-dimensional Euclidean space can be labelled by use of Cartesian (x, y, z), or cylindrical polar, or spherical coordinates, or a zillion other ways. None of these choices affects the Euclidean geometry. This choice is not what's at issue with the idea of a reference frame. For theoretical discussions, it is therefore usual to describe a reference frame by use of Cartesian coordinate systems, since those are generally easier to work with.

Choice of a reference frame can then be taken to mean a choice of Cartesian coordinate system for each different time t. There are many such choices, of course. Given our basic postulates, it does not make any sense to ask whether a reference frame is moving or rotating. It *does* make sense to ask whether a reference frame is moving or rotating *relative to* a second. Reference frames which are actually used for something are not chosen arbitrarily, of course, and often bear a special relationship to some object or person (which is stationary and non-rotating in that reference frame), but that is not required. A reference frame in principle says how to match up "the same" spatial positions at different times. However, since the Euclidean structure of space must be respected, a much simpler description is possible. If one reference frame is at hand, any other can be fully specified by saying how it's origin is moving and how the cartesian axes are rotating. That makes things rather simpler.

1.2.3 The Principle of Relativity

Hopefully by this time you've become rather agitated and want to scream out that some reference frames are good and some are evil because they're inhabited by 'fictitious forces' and other things too horrible to mention. That is true, but it answers a new question. The question is:

Is there a special class of reference frames singled out by dynamics?

The **Principle of Inertia** asserts that there exists a special class of reference frames, called **inertial reference frames** in which a particle not subjected to outside influences (i.e., forces) travels always at constant velocity. This is the same thing as Newton's 1st Law, but with this name it is usually associated with Galileo Galilei. Another principle comes hard on the heels of the Principle of Inertia. It asserts that we can go no further in winnowing down the set of reference frames to 'good' ones.

Principle of Relativity

The laws of physics are the same in any inertial reference frame.

This is a good statement of the Principle, but there's some nagging dissatisfaction with the wording. Very succinct statements of principle beg for paraphrases. Let's try a couple.

- No experiment can detect any intrinsic distinction between one inertial reference frame and another.

- 'Absolute rest' is not a physically relevant concept.[3]

[3] But very likely you will be more alert the next day if you get some.

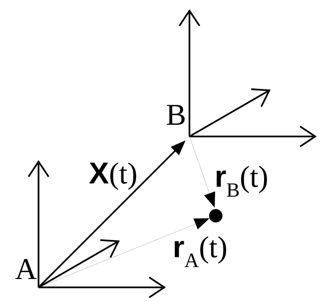

Figure 1.1: A pair of reference frames, non-rotating with respect to each other, but with non-zero relative velocity.

1.2.4 Galilean Transformations

So how is the description of motion in one inertial reference frame related to that in another? Indeed, how for any two reference frames whatever? Let reference frame A be given, and suppose the motion of a particle to be described by the trajectory $t \mapsto \vec{r}_A(t)$. This means that for each instant of time the coordinates of the particle with respect to A are given, and the subscript keeps track of the fact that this is position in A. As noted earlier, a second reference frame, B (naturally!) is specified by where its origin is as viewed from A, denoted by $\mathbf{X}(t)$ and how its axes are rotated, indicated by $R(t)$. First, take the special case $\mathbf{X}(t) \equiv 0$. Actually manipulating time-dependent rotations in three-dimensions involves some complications, something we put off to a later chapter. For now, let us simply observe that if a particle is stationary in frame A, it is certainly not stationary in frame B if it is rotating with respect to A. However, if the axes of B were rotated with respect to those of A, but their relative orientation were not changing, then the particle would be stationary also in B.

Having examined rotations, put them aside for the moment, and suppose that B's axes are always aligned with those of A. Now there is only $\mathbf{X}$ to play with. (See Figure 1.1) The relation between the particle's location in frame B and A is

$$\mathbf{r}_B(t) = \mathbf{r}_A(t) - \mathbf{X}. \tag{2.1}$$

If that sign looks funny to you, go back and check the definition of $\mathbf{X}$. Taking a time derivative[4],

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} - \frac{d\mathbf{X}}{dt}. \tag{2.2}$$

The left hand side is the velocity in B, which is therefore constant only if the relative velocity of frame B with respect to A, namely $\mathbf{V} = d\mathbf{X}/dt$, is constant. B is also an inertial reference frame only if it is moving at constant velocity with respect to A.

Finally, we notice that a constant shift of the spatial origin or of the zero of time does not affect velocities at all, so those changes will also preserve the inertial property.

Now take inventory. The set[5] of transformations of a reference frame which preserves the inertial character consists of:

- Constant shifts in space, $\mathbf{r} \mapsto \mathbf{r} + \mathbf{X}$,
- Constant shifts in time, $t \mapsto t + c$,
- An unchanging rotation, $\mathbf{r} \mapsto R \cdot \mathbf{r}$, and
- A shift of position linear in time, $\mathbf{r} \mapsto \mathbf{r} + \mathbf{V}t$.

These transformations can be applied succesively to build up more complicated transformations. These are called **Galilean transformations**.

Since the laws of physics are the same in two inertial reference frames, the actual history recorded in frame B could equally well have been observed in A, if the initial conditions had been different. This suggests a different way of using the Galilean transformations. We read them not as passive, relating two different descriptions of the same history, but as active. Applying the transformation to the initial conditions of an entire isolated system, rotating them or boosting the velocity of all the particles, the subsequent history is exactly the same as 'before' but with the same transformation applied.

Question Can you identify the assumptions about space and time which underlie the Aristotelian-Ptolemaic view of the world? What sorts of dynamical laws would be consistent with that?

1.3 Other Concepts of Single-Particle Dynamics

One way of phrasing the principle of Inertia is to say that a particle's momentum is **conserved** if there is no force acting upon it. There are some other concepts of single-particle mechanics whose value lies largely in their connection with such conservation laws. You are already familiar with these ideas. Let's recall them now.

[4]Notice that there's no problem in doing this since we have postulated universal agreement on the passage of time.

[5]If you are familiar with the concept, you may be interested to know that this set of transformations forms a **group**, in the mathematical sense.

1.3.1 angular momentum

First, if we choose an origin in an inertial reference frame, so that $\mathbf{r}$ is the vector from the origin to our particle, we define the **angular momentum** as

$$\mathbf{L} \stackrel{def}{=} \mathbf{r} \times \mathbf{p}. \tag{3.3}$$

This is a vector and it depends upon the choice of origin. Taking the cross product of $\mathbf{r}$ with each side of Newton's 2nd law,

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}. \tag{3.4}$$

The right hand side,

$$\mathbf{N} \stackrel{def}{=} \mathbf{r} \times \mathbf{F}, \tag{3.5}$$

is called the **torque** produced by the force $\mathbf{F}$. So, immediately you see that if the torque vanishes, the angular momentum of the particle is conserved. Requiring $\mathbf{N} = 0$ is a weaker condition than $\mathbf{F} = 0$, so this tells us something a little bit new compared to Newton's first law.

Exercise Think of some examples where $\mathbf{F} \neq 0$, but $\mathbf{N} = 0$. Does your experience verify constancy of $\mathbf{L}$?

1.3.2 kinetic energy, conservative forces and potentials

The **kinetic energy** of the particle is

$$T \stackrel{def}{=} \frac{m}{2}|\mathbf{v}|^2 = \frac{|\mathbf{p}|^2}{2m}. \tag{3.6}$$

If the particle suffers a displacement $d\mathbf{r}$ while the force $\mathbf{F}$ is acting, we say that the force does an amount of **work** given by

$$dW = \mathbf{F} \cdot d\mathbf{r}. \tag{3.7}$$

The value of this concept is largely due to the fact that the total work done on the particle is equal to the change in its kinetic energy, i.e.,

$$dT = \mathbf{F} \cdot d\mathbf{r}. \tag{3.8}$$

This is proved by differentiating T, inserting Newton's 2nd law and multiplying through by dt.

Exercise Make that computation.

Notice that we defined T directly. If you know the velocity, you know T and you can find its differential dT. By contrast, we did not define W itself, but rather dW. Is there a function W such that its differential really is $\mathbf{F} \cdot d\mathbf{r}$? There is no obvious reason why there should be. Let's see what that would entail. For now, we consider only forces which are given as functions of position, $\mathbf{F} = \mathbf{F}(\mathbf{r})$, and suppose that the elusive function $W(\mathbf{r})$ is really there. Then

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz,$$

which implies

$$\mathbf{F} = \nabla W(\mathbf{r}).$$

When the force does in fact come from a function in this the function $-W$ is called a **potential**. For some bizarre reason, potentials are generally denoted by U or V, so we write

$$\mathbf{F} = -\nabla V(\mathbf{r}). \tag{3.9}$$

Potentials are determined only up to an additive constant because such a constant will have no effect on the derivative.

When this situation pertains, Eq. (3.8) tells us that when the particle moves, the change of T is exactly equal to the change in $-V$, so that

$$\frac{dE}{dt} = \frac{d}{dt}(T + V) = 0. \tag{3.10}$$

We have a new conservation law. The value of the function $V(\mathbf{r})$ at the actual position of the particle is called its **potential energy**. Thus the total energy, $E = T + V$ given by adding potential and kinetic energies is conserved. Another nice thing about this is that the symbol for total energy makes sense. Because it results in conservation of the total energy, a force which derives from a potential in this manner is called a **conservative force**.

Is there a criterion on a force field $\mathbf{F}(\mathbf{r})$ which guarantees that it is conservative? Yes. The condition is

$$\nabla \times \mathbf{F}(\mathbf{r}) = 0. \tag{3.11}$$

To see how this works, suppose that you are given a force field and do not know whether it comes from a potential. You try to construct one by integrating Eq. (3.9):

$$V(\mathbf{r}) - V(\mathbf{0}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}. \tag{3.12}$$

If the force comes from a potential, this must hold for any path between the origin and $\mathbf{r}$. Referring to Figure 1.2, the integral forward along curve A or backward along B (notice the arrows) must give the same result. Thus, integrating forward along A and then B should give zero, i.e.,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0. \tag{3.13}$$

If this condition is met, we can consistently define the potential by integrating, if it is not, no potential exists. Applying Stokes' theorem to the integral and then taking the area enclosed by the loop to zero, we get the condition in Eq. (3.11). An example of a non-conservative force field is shown in Figure 1.3.

Question Would it make sense to have a potential function which also depended upon time? Upon velocity?

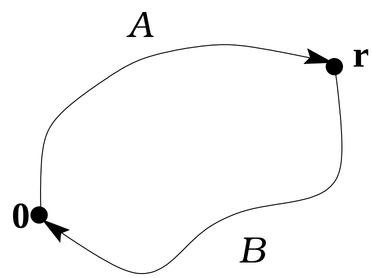

Figure 1.2: If a potential exists for $\mathbf{F}$, the line integrals along A and $-B$ (minus means against the arrow) must give the same answer. But then the integral along A and then along B must give zero.

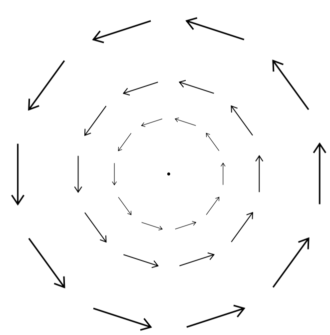

Figure 1.3: A non-conservative force field.

1.4 Systems of Particles

There's only so far we can get with a dynamics applicable to only a single particle, and it's not very far! In classical mechanics, a "particle" is anything that we intend to treat as though it were localized at a point and had no internal structure. Though not always, it is often legitimate to treat a macroscopic object as a particle. (Sometimes very big objects – in celestial mechanics, planets are often treated as point particles!) In so doing, we ignore completely the internal forces which hold the object together, and apply Newton's Laws as though there were only external forces acting on the body. We need to know that it's OK to do that. In this section, we establish some familiar (I hope!) results which say roughly that a multi-particle system or macroscopic body behaves mostly as though it were a point particle located at its center-of-mass. These results are Very Good Things, from a practical and theoretical perspective.

Without them, we would have to take the notion of 'particle' very seriously indeed. It would be necessary to identify the real particles to which Newton's Laws applied and then painfully build up to large objects, all the while keeping careful track of what the internal forces were doing.

Exercise Could a system of mechanics in which such a procedure was necessary make any sense at all? I'm not sure.

1.4.1 Center of Mass

So let's consider a system of N particles (N is my favorite number). We will use Greek letters to label the individual particles of the system. There are two basic quantities which characterize this system, and they tell us how much stuff there is and where it is. First, the total mass of the system is the sum of the masses of the constituent particles,

$$M = \sum m_\alpha. \tag{4.14}$$

Given that, we can define the **center-of-mass** of the system, which is sometimes abbreviated to 'COM' or even 'CM'. It is located by the vector

$$\mathbf{R}_{\text{CM}} \stackrel{def}{=} \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha, \tag{4.15}$$

where $\mathbf{r}_\alpha$ is the location of particle α. This is the "average" location of the mass of the system.

Question How would you actually go about measuring these things?

Often we deal with macroscopic objects which are neither legitimately particles themselves, nor seem to be easily resolvable into anything one might call particles. This situation is dealt with by successive approximations. Chop it up into pieces, apply the definitions as best as you may, subdivide the pieces, and continue *ad infinitum*. This limit process defines a mass density $\rho(\mathbf{r})$ such that the infinitesimal volume $dx\,dy\,dz$ has mass $\rho(\mathbf{r})dx\,dy\,dz$. Then the analogues of the above two formulas are

$$M = \int \rho(\mathbf{r})\, d^3r, \tag{4.16}$$

and

$$\mathbf{R}_{\rm CM} = \frac{1}{M}\int \rho(\mathbf{r})\mathbf{r}\, d^3r. \tag{4.17}$$

In this section, we will only work with the case of a finite number of discrete particles.

Exercise Transcribe the results to the continuum case, for your own amusement.

Here is a summary of the results to be established:

1. The time rate of change of the total momentum of a system of particles is equal to the sum of the external forces acting on it: $\dot{\mathbf{P}} = \mathbf{F}^{\rm ext}$. (Eq. 4.21)
2. The time rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques acting on it: $\dot{\mathbf{L}} = \mathbf{N}^{\rm ext}$. (Eq. 4.27)
3. The total kinetic energy of a system of particles is equal to the kinetic energy of a particle with mass M and moving with the center of mass, plus the kinetic energies of the individual particles as computed from their motion with respect to the COM. (Eq. 4.30)
4. If the internal forces are conservative, the time rate of change of the total energy of a system of particles is equal to the work done by external forces. (Eq. 4.33)

Actually, I've lied slightly. The result on angular momentum requires the so-called *strong form* of Newton's 3rd Law, as opposed to the weak form.

- Weak form: $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$. This is the form stated before.
- Strong form: Not only that, but $\mathbf{f}_{\alpha\beta}$ acts *along* the direction from α to β.

Also, the final result requires that the internal forces be derived from a potential. Neither of these is really a serious assumption, so we won't lose sleep over it. When they do not hold, there are other "degrees of freedom" to be taken into account, such as electromagnetic fields. Doing so makes it all better again.

1.4.2 Momentum

The total momentum of the system is given by

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}_{\rm CM}. \tag{4.18}$$

The first equality here ($\mathbf{p}_\alpha$ is the momentum of particle α) is simply a definition.

We are going to express the time derivative $d\mathbf{P}/dt$ in terms of forces. To do that, we split the force $\mathbf{F}_\alpha$ which acts on particle α into an external force $\mathbf{F}_\alpha^{\rm ext}$, and internal forces $\mathbf{f}_{\alpha\beta}$ which are due to other particles (β) in the system:

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{\rm ext} + \sum_\beta \mathbf{f}_{\alpha\beta}. \tag{4.19}$$

$\mathbf{f}_{\alpha\alpha} = 0$, as actually follows from the Principle of Inertia, so that we can include $\beta = \alpha$ if it's convenient, which it is. Feeding this into Eq. 4.18 *via* Newton's 2nd Law,

$$\frac{d\mathbf{P}}{dt} = \sum_\alpha m_\alpha \ddot{\mathbf{r}}_\alpha = \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} + \sum_{\alpha,\beta} \mathbf{f}_{\alpha\beta}. \tag{4.20}$$

However, the last sum vanishes, even under the weak form of the 3rd law. Finally then,

$$\frac{d\mathbf{P}}{dt} = \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} = \mathbf{F}^{\text{ext}}, \tag{4.21}$$

where this defines the total external force $\mathbf{F}^{\text{ext}}$.
Exercise Fill in the gaps.

1.4.3 Angular Momentum

You can make The total angular momentum $\mathbf{L}$ is expressed in terms of the angular momenta $\mathbf{L}_\alpha$ of the individual particles in a similar way:

$$\mathbf{L} = \sum_\alpha \mathbf{L}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{4.22}$$

It is useful to split the positions and velocities into those of the center of mass, plus relative positions and velocities. We write

$$\mathbf{r}_\alpha = \mathbf{r}'_\alpha + \mathbf{R}_{\text{CM}}, \tag{4.23}$$

and

$$\mathbf{v}_\alpha = \mathbf{v}'_\alpha + \mathbf{V}_{\text{CM}}. \tag{4.24}$$

Substituting these expressions into the sum of angular momenta, you will find

$$\mathbf{L} = \sum_\alpha \mathbf{L}_\alpha = \mathbf{R} \times \mathbf{P} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha. \tag{4.25}$$

Applying Newton's 2nd Law to the individual particles,

$$\frac{d\mathbf{L}}{dt} = \sum_\alpha \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} + \sum_{\alpha,\beta} \mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}. \tag{4.26}$$

In the final expression of Eq. 4.27, the first term $\sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} = \sum_\alpha \mathbf{N}_\alpha^{\text{ext}} = \mathbf{N}^{\text{ext}}$ is the sum of the external torques; that is what we want, in case you've forgotten. However, we would really like to get rid of the second term. If we can do that, the announced result is left,

$$\frac{d\mathbf{L}}{dt} = \sum_\alpha \mathbf{N}_\alpha^{\text{ext}}. \tag{4.27}$$

To get there, use the separation vector

$$\mathbf{r}_{\alpha\beta} \equiv \mathbf{r}_\alpha - \mathbf{r}_\beta$$

between particles α and β. Here's where the strong form of the 3rd Law comes in. It implies that $\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta} = 0$. The last term on the right-hand side of Eq. (4.26) can now be shown to vanish:

$$\sum_{\alpha,\beta} \mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta} = 0. \tag{4.28}$$

Exercise Fill in the gaps in all these manipulations. Some cross terms which you might have expected in Eqs. (4.25) and (4.26), such as $\sum \dot{\mathbf{r}}_\alpha \times \mathbf{p}_\alpha$, vanish because of the definition of center of mass. For Eq. (4.28), use the remark to substitute $-\mathbf{r}_\beta$ for $\mathbf{r}_\alpha$, and also remember that $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$.

1.4.4 Energy

By now, the routine must be familiar. Write the total kinetic energy as the sum of kinetic energies of the particles in the system:

$$T = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\mathbf{r}}_\alpha|^2. \tag{4.29}$$

As before, make the decomposition $\mathbf{r}_\alpha = \mathbf{R}_{\mathrm{CM}} + \mathbf{r}'_\alpha$, to find

$$|\dot{\mathbf{r}}_\alpha|^2 = |\dot{\mathbf{R}}|^2 + 2\dot{\mathbf{R}} \cdot \dot{\mathbf{r}}'_\alpha + |\dot{\mathbf{r}}'_\alpha|^2.$$

Put this back into Eq. 4.29, and convince yourself that the sum of cross terms between $\dot{\mathbf{R}}$ and $\dot{\mathbf{r}}_\alpha$ vanishes (Hint: differentiate the definition of COM with respect to time). That leaves

$$T = \frac{1}{2} M |\mathbf{V}|^2 + \sum_\alpha \frac{1}{2} m_\alpha |\dot{\mathbf{r}}'_\alpha|^2, \tag{4.30}$$

which is one of the results announced at the beginning of this section.

To get the final result, we must show that the work done by internal forces is offset by a loss of potential energy. The work done by the forces acting between particles α and β when they undergo an infinitesimal displacement is

$$dW_{\alpha\beta} = \mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_\alpha + \mathbf{f}_{\beta\alpha} \cdot d\mathbf{r}_\beta = \mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_{\alpha\beta}. \tag{4.31}$$

At this point we must assume that the forces are given by a potential. No surprise there. We also expect that this force should derive from a potential depending only upon the *relative* positions of particles α and β. (Shifting them both a bit to the west should not affect their interaction!) So,

$$\mathbf{f}_{\alpha\beta} = -\nabla_\alpha \overline{U}_{\alpha\beta}(\mathbf{r}_{\alpha\beta}) = -\nabla \overline{U}_{\alpha\beta}(\mathbf{r}_{\alpha\beta}). \tag{4.32}$$

In the second expression, the subscript on the ∇ operator indicates differentiation with respect to $\mathbf{r}_\alpha$ with $\mathbf{r}_\beta$ held fixed. Since U is really a function only of $\mathbf{r}_{\alpha\beta}$, this is traded for a differetiation with respect to that variable in the final expression. The work in Eq. (4.31) is therefore equal to $-d\overline{U}_{\alpha\beta}$. Due to the force acting between them, the kinetic energy of particles α and β is increased by dW, but they suffer a loss of potential energy which exactly compensates. We are left with the result

$$\frac{dE}{dt} = \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} \cdot \dot{\mathbf{r}}_\alpha. \tag{4.33}$$

Question What if the potentials depended upon three or more particle positions?

1.5 Conservation Principles

Newton's Laws imply that for any mechanical system whatsoever, certain quantities are conserved. These are

1. Energy
2. Momentum
3. Angular Momentum.

We have derived these conservation theorems from Newton's Laws, although an additional assumption was required for the first. However, it is worth emphasizing that they have a validity beyond the Newtonian framework. That is why I prefer to call them *principles* rather than theorems, as MT do. Later, in studying Lagrangian mechanics, we will see how they arise from some very fundamental properties of space and time. On that basis, they extend far beyond mechanics. Conservation principles also serve a couple of practical purposes in mechanics. They allow us to get partial information about dynamics when a complete solution is too difficult. They also may be used to prepare the ground for a detailed solution of the equations of motion, helping us to choose the correct variables. The Kepler problem, studied in chapter 5 (MT Ch. 8) is a splendid example of this.

1.6 The Two Body Problem

To alleviate the abstraction, this section is somewhat more concrete. We make our first foray into the two body problem. we will see how it effectively reduces to a one-particle by use of the ideas of the previous sections. We also examine the relation between scattering angles in the center of mass frame and a frame in which

Our object of study is a pair of particles isolated from the rest of the world. The center of mass is

$$\mathbf{R} = \frac{1}{M}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right). \tag{6.34}$$

As in section 1.4, the positions are split into the center of mass position plus the position of the particles with respect to the center of mass:

$$\begin{aligned} \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R}, \\ \mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R}. \end{aligned} \tag{6.35}$$

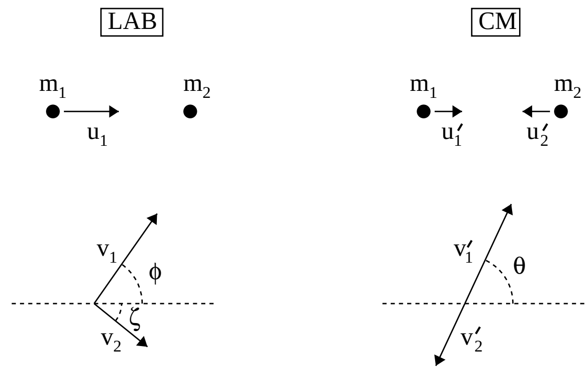

Figure 1.4: A scattering experiment as viewed from two inertial reference frames. One is the laboratory frame (LAB) in which rapidly moving particle 1 collides with the stationary particle 2. The other is the center of mass (CM) frame, in which their momenta are equal and opposite at all times, so that there is only one scattering angle in CM.

These can be solved explicitly for the primed positions in terms of the relative displacement from 2 to 1. The result is

$$\mathbf{r}_1' = \frac{m_2}{M}\mathbf{r}, \qquad \mathbf{r}_2' = -\frac{m_1}{M}\mathbf{r}, \tag{6.36}$$

where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2'. \tag{6.37}$$

is the position of particle 1 relative to particle 2, i.e., their separation vector.

We don't need to worry about $\mathbf{R}$. The only question is how $\mathbf{r}$ changes with time. But this is directly computed to be

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{f}_{12}}{m_1} - \frac{\mathbf{f}_{21}}{m_2} = \frac{\mathbf{f}}{\mu}. \tag{6.38}$$

The **reduced mass** μ here is defined by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \tag{6.39}$$

and the force $\mathbf{f}$ is $\mathbf{f}_{12}$, the force exerted by particle 2 on particle 1.

Exercise Fill in a bit of algebra.

The dynamics of the pair of particles, in the CM frame, is essentially equivalent to that of a single particle with mass μ and acted upon by a force $\mathbf{f}$. To make further progress we need to know about the force $\mathbf{f}$. But this does point out the intrinsic interest of the center of mass frame.

Sometimes, at the subatomic level, interactions are probed by means of scattering experiments. But these are often carried out in a laboratory in which one particle is a stationary target and the other is a high-energy projectile. Hardly the center of mass frame, there! See figure 1.4 for a schematic of the geometry in the LAB and CM frames. If we want to predict scattering angle ϕ, which is the directly measured quantity and belongs to the LAB frame, or infer from measured scattering something about the interaction, we want to know how to relate the angles ϕ and ζ to the scattering angle θ in the CM frame.[6] We'll also assume that the scattering event is elastic, so that total kinetic energy of the particles is conserved.

The total momentum of the pair in the LAB frame is

$$\mathbf{P} = m_1 \mathbf{u}_1. \tag{6.40}$$

By throwing some masses into formulas you've seen before, the momenta in the two frames can be related by

$$\mathbf{p}_i = \mathbf{p}'_i + m_i \mathbf{V} = \mathbf{p}'_i + \frac{m_i}{M}\mathbf{P}. \tag{6.41}$$

Taking the dot product of $\mathbf{P}$ with $\mathbf{p}_{1,f} = m_1 v_1$ (the final momentum of particle 1), we get the cosine of ϕ, and by taking the magnitude of their cross product, we get $\sin\phi$. Actually, we also have the magnitudes $|\mathbf{p}_{1,f}|$ and $|\mathbf{P}|$ in the results if we do that, but they will cancel out of the ratio. So, go for the tangent:

$$\tan\psi = \frac{|\mathbf{p}_{1,f} \times \mathbf{P}|}{\mathbf{p}_{1,f} \cdot \mathbf{P}}. \tag{6.42}$$

Just express $\mathbf{p}_{1,f}$ in terms of the CM quantities and you'll be done.

To make that easier, observe that the kinetic energy in the CM frame is

$$T' = \frac{1}{2m_1}|\mathbf{p}'_1|^2 + \frac{1}{2m_2}|\mathbf{p}'_2|^2. \tag{6.43}$$

But, $\mathbf{p}'_2 = -\mathbf{p}'_1$. Conservation of energy then implies that the magnitudes of the momenta in CM are unchanged by the scattering: $|\mathbf{p}'_{1,f}| = |\mathbf{p}'_{1,i}$.

Now we can plug the relation of Eq. (6.41) into Eq. (6.42), and a bit more algebra yields the final result, which is

$$\tan\psi = \frac{m_2 \sin\theta}{m_1 + m_2\cos\theta}. \tag{6.44}$$

1.7 Conservative Versus Dissipative Dynamics

Not all forces are conservative, as you well know. The assumption of no friction for various problems in elementary mechanics is an eternal source of amusement for students.

[6]MT work this through using velocities (§9.7). We will do it with momenta.

1.7.1 some empirical formulas

Two planar solid surfaces in contact, forced together by a normal force N is an abstraction of a common situation. The normal force is often the weight of one of the objects in contact. If another force is applied tangent to the surface of contact, as to impel sliding of the surfaces against each other, a frictional force, just sufficient to counter it arises, up to a maximum value of

$$F \leq \mu_s N, \tag{7.45}$$

where μ_s, known as the *coefficient of static friction* is an empirical constant depending upon the nature of the two surfaces. If this friction is overcome so that there is sliding, a residual frictional force exists, given approximately by

$$F = \mu_k N, \tag{7.46}$$

where μ_k is the coefficient of *kinetic* friction. Again, μ_k depends upon the surfaces, but always $\mu_k < \mu_s$.

Question Why must this inequality hold?

Another very common sort of dissipative force arises when objects move through a fluid such as the air, or water. This is a very different situation because the fluid does not move rigidly against the body, as with solid-on-solid motion. Generally, the fluid *does not* slip against the surface. Rather (at low speeds), velocity gradients are set up in the fluid, and energy is dissipated through viscosity as layers of fluid moving at different speeds try to force neighboring layers into lockstep. The magnitude of the resulting force on the body has a velocity dependence as a result. Often, the force is adequately approximated by

$$\mathbf{F} = -c\mathbf{v}, \tag{7.47}$$

where c is a constant depending upon the fluid and the body. The minus sign indicates that the force is directed opposite the velocity of the body relative to the fluid. The constant c can be calculated for a spherical object of radius a, with the result

$$c = 6\pi a\eta, \tag{7.48}$$

where η is the viscosity of the fluid. For other bodies, one anticipates that c is roughly proportional to a typical linear dimension.

At higher speeds of motion through a fluid, the flow set up in the fluid is no longer smooth. Eddies and turbulence start to develop and it is much more difficult to calculate in this regime. However, a rough-and-ready formula is

$$F = CSv^2, \tag{7.49}$$

where the constant C again depends upon fluid properties and S is the cross-sectional area that the body presents to the fluid.

Question In section 1.3, a test was given for checking whether force fields were coservative. Why is this test not even applicable to the forces described here? Are the solid-on-solid frictional forces velocity dependent? How so?

1.7.2 the problem of dissipation

Checking whether a force field is conservative by computing the curl (? $\nabla\times\mathbf{F}=0$?) requires that the force actually be a function of position only. Many real-life dissipative forces, like the ones described here, are velocity dependent. But not all velocity-dependent forces are dissipative. The Lorentz force of a magnetic field on a charged particle, $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$ is an example of a conservative velocity dependent force.

Exercise Prove that.

We have good reason to believe very strongly in the conservation of energy under all circumstances. That being so, one must conclude that dissipative forces are a sort of illusion, arising from ignoring some of the places energy could go. Generally it shows up as heat, motion of molecules on a microscopic level. Sometimes it is as even less mechanical forms of energy such as radiation. Either way, the dissipation is not fundamental. But it's not convenient, or even possible, to keep track of all the microscopic motions or the radiation. So it's very hard to understand dissipative forces in detail. Their being sort of a 'garbage' phenomenon has perhaps also led to an attitude that they aren't worth studying.

From the perspective of solving dynamics, dissipative forces either make things very hard, or very easy. If their is no input of energy from outside, no driving, a dissipative system quickly settles down to an equilibrium. So that's the easy. If there is a driving, though, it is very hard. This is because forces which do not conserve energy probably don't conserve much of anything else, angular momentum for instance, and these conservation laws are often quite important in obtaining solutions.

Question But linear momentum is different. Why?

1.8 Gravitation

> That one body may act upon another at a distance through a vacuum, without the mediation of any thing else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty of thinking can ever fall into it.
>
> Isaac Newton in a 1692 letter to Richard Bently

1.8.1 historical musings

Currently, depeding upon how you count, we know of four fundamental forces. These are gravity, electromagnetism, and the weak and strong nuclear forces. The last two act only over such short distances that quantum mechanics is essential to their operation. A purely classical description isn't really possible (though it can provide a useful starting point). Just as electricity and magnetism have been understood as aspects of a single phenomenon since the time of Maxwell, electromagnetism and the weak force now fit into a unified quantum mechanical framework. Much effort has been devoted in the past quarter-century to 'Grand Unification' of these with the strong force, but a satisfactory result has not been achieved. Albert Einstein spent the last two or three decades of his life in a futile effort to unify gravity with electromagnetism. Unifying gravity with the other forces, indeed even obtaining a consistent quantum-mechanical theory of gravity itself remains something of a holy grail for many theoretical physicists. It may seem rather ironic that this should be the fate of the fundamental force which was first recognized and understood to any extent. However, as Newton recognized, he didn't really have a theory of gravity at all. He had a theory of the way gravity affects matter. That is completely different. The question of possible dynamics of gravity itself, of the whatever-it-is that transmits the force from here to there, is ignored completely. But you've got to start somewhere, and that Newton did, with one giant leap.

The law which Newton proposed for the gravitational force exerted by a body of mass M on a second body of mass m is

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\hat{\mathbf{r}}, \tag{8.50}$$

where $\mathbf{r}$ is the vector from the location of M to that of m, and G is Newton's gravitational constant, with the value

$$G = 6.6672 \times 10^{-11}\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2}. \tag{8.51}$$

Henry Cavendish made the first determination of this universal constant by measuring the forces between masses of a few kg! If that's not impressive, I don't know what is.

Exercise Go find out about the Cavendish experiment.

There are a couple of weird things about the gravitational force as we've set it up here. The first is the thing which bothered Newton. How does the force get from here to there? Fortunately, he seems to have realized that that question had to be left to another age. The other thing you might notice is the way the masses enter the force law. Recall the Coulomb law for the electric force between a pair of electric charges, and you will see that the masses are entering here in the rôle of *gravitational charges.* The mass is first introduced into mechanics as a proportionality between the force acting on a body and the acceleration that body suffers as a result. So it determines *response* to a force. Why should it also determine the *value* of the force? After all, there is no relation whatever between the mass of a body and its electric charge. This second observation was key in Albert Einstein's thinking that led to the General theory of Relativity. It was only with the latter theory, arriving on the scene in 1915, that the first puzzle was really cleared up. Alas, we cannot go into these these matters here. We content ourselves with taking the force law as given and see what we can do with it.

1.8.2 gravitational field and potential

Notice that the gravitational force in Eq. (8.50) can be written as

$$\mathbf{F} = m\mathbf{g},$$

where

$$\mathbf{g}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}}. \tag{8.52}$$

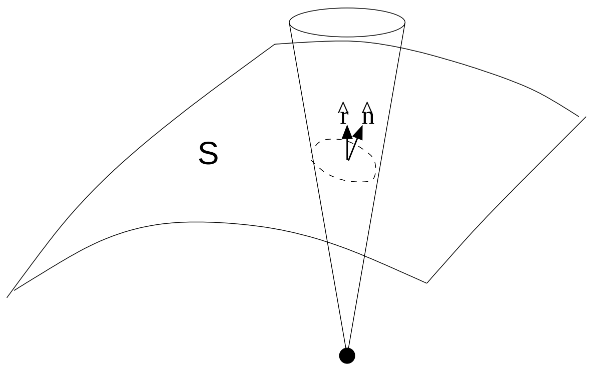

Figure 1.5:

Since **g** does not depend upon the body being acted upon, but only the one producing the force, one is inclined to consider this vector as a more fundamental quantity. It is called the **gravitational field**.

As noted in the last section, a central force is conservative and therefore can be expressed in terms of a potential. This will also be true of the gravitational field and we can express it as

$$\mathbf{g}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}), \tag{8.53}$$

in terms of the **gravitational potential**

$$\Phi(\mathbf{r}) = -\frac{GM}{r}. \tag{8.54}$$

You will have noticed that I treated the masses involved as if they were located at points. This is not generally true, of course, and the extended nature of gravitating bodies can matter. Suppose we wish to compute the gravitational force exerted by the earth on the moon. The moon is not so terribly far away. Maybe we should not pretend that either it or the earth is point-like. In fact, this was a problem which worried Newton in the twenty years between finding the gravitational force law and publication of the *Principia*. As we'll see in a bit, for spherically symmetric bodies, it actually doesn't matter!

Coming to grips with the fact that real massive objects are extended, it would be better to describe a body by means of a **mass density**

$$\rho(\mathbf{r}),$$

such that the mass in a volume $dx\,dy\,dz$ is $\rho\,dx\,dy\,dz$. Then the gravitational potential produced at point **r** by all this stuff is

$$\Phi(\mathbf{r}) = -G\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'. \tag{8.55}$$

Exercise Write the analogous expresion for a number of discrete point-like masses. Write a similar integral for 'adding up' the gravitational *field*, rather than the potential.

1.8.3 Poisson's equation

Now to the promised result. Define the **gravitational flux** through the surface S to be the integral

$$\text{Flux}(S) = \int_S \mathbf{g} \cdot \hat{\mathbf{r}}\, d^2 s. \tag{8.56}$$

The amazing thing is this: as long the surface S entirely encloses the mass, the flux is independent of its shape! As a formula,

$$\text{Flux}(S) = -4\pi GM \qquad \text{if } S \text{ is closed and the mass } M \text{ is inside.} \tag{8.57}$$

This will be proved for a point mass. The general result is then to be had by superposition. Figure 1.5 shows a point mass and part of a surface S through which the gravitational flux is to be computed. We compute the flux through the small area defined by the intersection of the cone with S, which is at a distance r from the point mass M. It is

$$\frac{d(\text{Flux})}{GM} = -\frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}\, dA}{r^2}, \tag{8.58}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to S and dA is the area cut out by the cone. But $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} dA$ is the area of the projection perpendicular to $\hat{\mathbf{r}}$, which is proportional to r^2 and independent of the angle at which S cuts across the cone. (The infinitesimal cone defines a fixed solid angle. At any distance from the point, the cross sectional area is a fixed fraction of a spherical surface area $4\pi r^2$.) The right hand side of Eq. (8.58) therefore actually depends only upon the cone. Thus, the magnitude of the flux is the same through *any* surface spanning the cone. The sign depends upon whether the unit normal $\hat{\mathbf{n}}$ points outward or inward, however.

Now cover all space with tiny cones emanating from the point. They can't all be *circular* cones, but that was not an essential feature anyway. If the point mass is outside the surface S, each cone will punch through S an even number of times (it may be folded over in intricate ways). It must go in, then out, then..., and finally wind up out. The outward unit normal vectors to the surface point toward the apex of the cone in one case and away in the other. The net flux through the surface is therefore zero.

If the mass is inside the surface, each cone will punch through an odd number of times. There is an inside-to-outside passage left over. The net flux associated with any cone is exactly the same as is the surface had been a sphere centered on the mass, because that surface too would have crossed each cone once with the right orientation of $\hat{\mathbf{n}}$. Computing the flux for a sphere is very easy, and the stated value results. ■

Exercise Become comfortable with this argument. Or invent another argument. One idea is to use *lines of force*. This notion was invented by Michael Faraday during his researches on magnetism and is really the germ of the (dynamical) field concept. You can reduce the theorem to a counting argument this way.

The transcription of the flux theorem into the potential language is also interesting. The statement,

$$\nabla^2 \Phi = 4\pi G\rho, \tag{8.59}$$

is known as **Poisson's equation**. Green's theorem is instrumental in demonstrating this. Starting from the flux theorem for a surface S enclosing a volume Ω,

$$\text{Flux}(S) = \int_S \mathbf{g} \cdot \hat{\mathbf{n}}\, d^2 s = -4\pi G M(\Omega), \tag{8.60}$$

where $M(\Omega)$ is the amount of mass contained in Ω, Green's theorem converts it into

$$-4\pi G M(\Omega) = \int_\Omega \nabla \cdot \mathbf{g}\; d^3 r,$$

where Ω is the volume enclosed by S. But, $\nabla \cdot \mathbf{g} = -\nabla^2 \Phi$. Shrinking the volume of Ω to zero, so that $M(\Omega) \to \rho(\mathbf{r})\text{vol}(\Omega)$,

$$4\pi G \rho(\mathbf{r}) = \nabla^2 \Phi.$$

Perhaps it is not yet clear why this means that the earth and moon may be treated as point masses. Let's tackle the body producing the field first; that's the earth in this case. The earth has a very nearly spherically symmetric mass distribution. This means that any rotation about the center will not alter the distribution of mass. The appearance is not spherically symmetric because a rotation moves where the various continents are, but that's irrelevant. As a result, the gravitational field has to have the form $f(r)\hat{\mathbf{r}}$ for some function $f(r)$. If it did not point radially inward or outward everywhere, a rotation would change it without changing the mass distribution causing it. Absurd! Now, using the theorem, we get $(4\pi r^2) f(r) = -4\pi G M$. Rearranging gives a field identical to that of a point mass located at the center.

That's half the game. Demonstrating that the total force on the moon is the same as if it too were concentrated at its center is trickier. Unless I'm missing something.

Exercise Try to prove that. Use Newton's 3rd Law and the fact that we've already shrunk the earth.

This result is very useful since many astonomical objects, apart from asteroids and alien spacecraft, are pretty close to having a spherically symmetric mass distribution.

1.8.4 Tides

What was proved about shrinking spherically symmetric bodies down for purposes of computing gravitational interactions applied only to the *total* force. It does not mean that, for instance, the gravitational pull of the moon (or the sun) on one side of the earth is the same as it is on the other. It is this difference which gives rise to the tides. Actually, the shapes of seafloors and inlets etc., affect how big tides are in different locations, sometimes quite dramatically, but that is the basic driving force.

The tides are very interesting and deserve some discussion, but they won't get it here. Please see MT, §5.5.

1.9 Beyond Newton's Mechanics

A pretty good discussion of the limitations of Newtonian mechanics and the modifications required in the structure by special relativity and quantum mechanics is found in MT §2.8. I will resist the temptation to babble about the subject, but will pose a

Question Why bother with classical non-relativistic mechanics if we know it is wrong?

1.10 Notes

This chapter draws on material which is scattered throughout MT. Elementary Newtonian mechanics, along with numerous example problems, is treated in MT Ch. 2. MT have relegated the consideration of systems of particles, including the basic results required for consistent application of Newtonian mechanics (discussed here in section 1.4) to Chapter 9, §1-7. The aspects of gravity treated in section 1.8 are found in MT Ch. 5, together with additional material on tides.