Chapter 2

Lagrangian Dynamics

2.1 Overview

This chapter presents the Lagrangian formulation of mechanics. The Lagrangian approach has practical advantages in that it allows equations of motion to be found much more easily than with the Newtonian techniques. It also makes possible new general insights about dynamics. This is not an abandonment of Newton, either in practice or principle. As far as Newtonian mechanics is valid, the two formulations are completely equivalent. Further, we will continue to use Newton's Laws directly. The Langrangian formalism does extend outside of mechanics, however, and in that sense, it contains more. Section 2.3 deduces the new form of equations of motion from Newton's Laws. Then some examples are given which illustrate the Lagrangian procedure for finding equations of motion. Later, we see how the equations of motion come from a new principle (Hamilton's Principle) by use of techniques from the calculus of variatons. An introduction to that mathematics is given in section 2.5 and 2.6. Before any of that, however, some groundwork is needed concerning the spaces used to describe dynamics. These tools, configuration space and phase space, are also useful in other contexts.

2.2 Configuration Space and Velocity Phase Space

2.2.1 generalized coordinates

In dynamics, we are concerned not only with where things *really are in a given situation*, but also with where things *could be* under other circumstances (initial conditions). For instance, a baseball might conceivably be anywhere in a three-dimensional space, or a hockey puck anywhere on a plane (on the proverbial frictionless pond — but not in August). With Cartesian coordinates, we would have x, y and z taking all real values, making up $\mathbb{R}^3$ in the first case, and all possible values of x and y making up $\mathbb{R}^2$ in the second. Of course, we don't really need unbounded spaces here, but cutting them down is rather inconvenient. The set of all *where things could be* is called the **configuration space**. For a system of N particles free to move throughout 3-space, we need three unrestricted coordinates for each

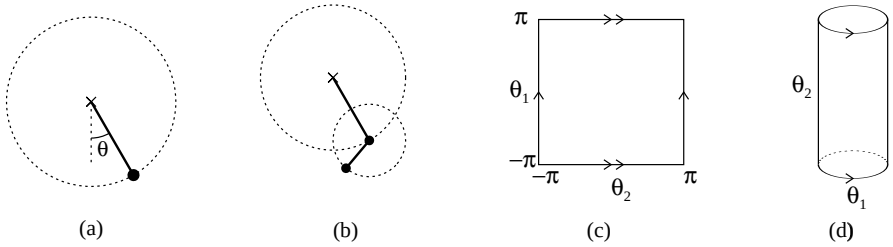

Figure 2.1: Configuration spaces. (a) is an ordinary pendulum on a rigid rod. The configuration space is simply a circle, which mathematicians denote by S^1. (b) is a double pendulum, made by attaching a second rod to the bob of the first so that it, too, swings freely in the plane. Now the configurations are specified by two completely independent angles. The possible pairs of angles make up the square in (c). But, since $\theta_1 = -\pi$ is the same as $\theta_1 = \pi$ and similarly for θ_2, the opposite sides should be identified arrow to arrow and double arrow to double arrow. Making one identification gives a finite cylinder with opposite ends identified. If we actually join those together, we will get something that looks like the surface of a doughnut, known as a torus (not to be confused with Taurus, which is made by Ford).

particle, making a total of $3N$. This configuration space can be identified with $\mathbb{R}^{3N}$.

For an example with a new twist, consider a (rigid) plane pendulum free to rotate all the way around its point of support. The accessible positions of the bob lie on a circle in this case. So, our configuration space is a circle, often denoted S^1 by mathematicians. What if it can swivel around the support in any direction? Then the bob can be anywhere on a sphere, so the configuration space is the surface of an ordinary sphere (S^2). (Of course, there is a slight engineering problem if *all* directions are to be accesible!)

Exercise What is the configuration space of the double pendulum in Figure 2.1(b)? Suppose you hold your wrist rigid and your shoulder is fixed in position, but free to swivel. What is the configuration space for your arm? Try to picture it.

Usually we need to use some sort of coordinates to specify particular points in the configuration space, such as the Cartesian coordinates mentioned for the baseball and the puck. The ease with which vectors can be handled in such coordinates makes them very attractive. Unfortunately, Cartesian coordinates are not always well-adapted to the problem at hand. Who can say what sort of coordinates we might need for a particular problem? The possibilities are boundless. Consider, for a moment, the problem of attaching coordinates to the plane. To make a legitimate coordinate system, all we need is a pair of smoothly varying functions q_1 and q_2 on the plane such that a pair of values (q_1, q_2) uniquely specifies a point. Figure 2.2 depicts a system of coordinates. This is probably a rather impractical system, but it illustrates the abundance of possibilities.

Newton's Laws are expressed in terms of vectors, so we like Cartesian coordinates. But some problems, the double pendulum for example, are more easily and naturally described using other coordinates. Perhaps this is a technical difficulty, and we'll just have to "deal

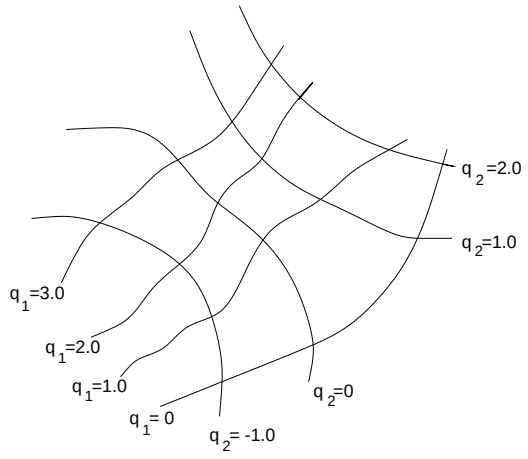

Figure 2.2: A rather unwieldy system of coordinates in the plane.

with it," swallowing one difficulty or the other. Actually, we are going to formulate mechanics in a new way which (a) allows arbitrary coordinates, even non-inertial ones, (b) doesn't really need coordinates at all, making some theoretical arguments very simple, (c) extends to the dynamics of systems far beyond what you might call mechanics, electromagnetic fields for example, and (d) comes in four designer colors.[1] Unbelievable? Read on.

2.2.2 velocity phase space and some kinematics

Time to get down to business, then, and start looking at some relations between different coordinate systems. Let there be N particles in your system, and denote their Cartesian coordinates in an inertial reference frame by x^i for $i = 1, 2, 3, \ldots, 3N$ (if they're confined to a plane, i only goes up to $2N$). Let q^j, $j = 1, 2, \ldots, 3N$ be another system of coordinates on the configuration space. Don't worry, I won't ask how you got them. There will be some relation between the two, indicated abstractly as

$$q^j = q^j(\{x^i\}, t), \tag{2.1}$$

where the braces indicate that the i on x^i is just a placeholder. We've also anticipated the possible desirability of having the relation between the q's and x's depend on time. That would be the case for example if the q's were rigidly attached to a rotating body. Though a nice formula might not be available, this relation can be inverted, in principle, to yield

$$x^i = x^i(\{q^j\}, t). \tag{2.2}$$

Since x^i is a function of the q's and t, and nothing else, its differential can be written as

$$dx^j = \frac{\partial x^j}{\partial q^i} dq^i + \frac{\partial x^j}{\partial t} dt. \tag{2.3}$$

[1] Subject to availability. Offer void where prohibited by law.

Dividing through by dt,

$$\dot{x}^j = \frac{\partial x^j}{\partial q^i}\dot{q}^i + \frac{\partial x^j}{\partial t}. \tag{2.4}$$

Since $x^j(q,t)$ does not depend upon $\dot{q}$, differentiation of this last relation yields

$$\frac{\partial \dot{x}^j}{\partial \dot{q}^i} = \frac{\partial x^j}{\partial q^i}. \tag{2.5}$$

This last relation is important because the x's alone are not enough to determine the future of the system, even if you also know the forces. You also need *velocities*. For each coordinate x^i (or q^i), there is a corresponding velocity, $\dot{x}^i$ ($\dot{q}^i$). Combining the configuration space coordinates with the velocities gives **velocity phase space**, though sometimes one says just "phase space." The simplest possible example is a particle moving in one dimension x. Then the velocity phase space is a plane, and we could plot position-velocity pairs with the y coordinate being the velocity. For a little more flavor, try the plane pendulum again. Plot the angle θ on a horizontal axis ("x") and the velocity $\dot{\theta}$ along a vertical axis ("y"). Remember that $\theta = -\pi$ is the same as $\theta = \pi$, so roll it up to make a cylinder. Since the angular velocity $\dot{\theta}$ could be arbitrarily large, there is no rolling up in that direction.

When we change from x's to q's on the configuration space, we must also switch $\dot{x}$'s for $\dot{q}$'s on the velocity phase space. But there are no new free choices (as there were with q). The relation between the two sets of velocities is locked in according to Eq. (2.5).

Notice that dynamics has not yet come into the picture. What we've done is really kinematics – coordinates and velocities. To do anything interesting will require some laws of motion, taken up in the next section.

But first, a relation which will be useful in a bit. The total time derivative of a function on velocity phase space is given by

$$\frac{d}{dt} = \sum_i \left(\ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \dot{q}^i \frac{\partial}{\partial q^i} \right) + \frac{\partial}{\partial t}.$$

Be careful about what this means. The derivative operator on the left is along the *actual motion* $(q(t), \dot{q}(t))$. On the right, we have partial derivatives with respect to $\dot{q}^i$ with all other $\dot{q}^j$'s, all the q^j's, and t held fixed, then with respect to q^i, with all the *other* q^j's, *all* the $\dot{q}^i$'s and t held fixed, and finally, with respect to t with all the phase space coordinates q and $\dot{q}$ held fixed. That is what will always be meant by those kinds of partial derivatives. Now apply this to $\partial x(q,t)/\partial q^i$ to get

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = \left[\dot{q}\frac{\partial}{\partial q} + \frac{\partial}{\partial t}\right]\frac{\partial}{\partial q}x = \frac{\partial}{\partial q}\frac{dx}{dt}. \tag{2.6}$$

We'll need this equation in the next section.

Exercise Verify the relation of Eq. (2.6). You need to justify the apparent disappearance of one term from the expansion of d/dt, reverse the order of some partial derivatives and pack things back up into d/dt. (hint: on which variables in $\{q, \dot{q}, t\}$ does x depend? What is the value of $\partial\dot{q}/\partial q$?)

2.3 Rewriting Newton's Laws of Motion

Let's write Newton's second law in Cartesian components:

$$\frac{dp_i}{dt} = F_i. \tag{3.7}$$

The index i indicates which component. Sometimes it will be 'up', sometimes 'down.' The placement doesn't really mean anything for us. Just playing around a bit, observe that each side can be written as a derivative, since

$$p_i = \frac{\partial T}{\partial \dot{x}^i}, \tag{3.8}$$

and

$$F_i = -\frac{\partial U}{\partial x^i}. \tag{3.9}$$

Putting these together, we obtain an expression of Newton's 2nd law in the funny-looking form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) = -\frac{\partial U}{\partial x^i}. \tag{3.10}$$

2.3.1 one degree of freedom

It would be nice to be able to write Newton's equations of motion in a form which looks the same in any coordinate system. Perhaps this last equation generalizes. Let's drop back to just one degree of freedom for the time being. Then you might guess

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) \stackrel{?}{=} -\frac{\partial U}{\partial q}. \tag{3.11}$$

Checking what's in parentheses on the left, you find

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{x}}\frac{\partial x}{\partial q}, \tag{3.12}$$

which reduces to

$$\frac{\partial T}{\partial \dot{q}} = p\frac{\partial x}{\partial q}. \tag{3.13}$$

Eq. (2.5) was used to "cancel the dots".

Inserting this into the left-hand side of our guess,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}\left(p\frac{\partial x}{\partial q}\right) = \frac{dp}{dt}\frac{\partial x}{\partial q} + p\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right). \tag{3.14}$$

The first term on the right is

$$\frac{dp}{dt}\frac{\partial x}{\partial q} = -\frac{\partial U}{\partial x}\frac{\partial x}{\partial q} = -\frac{\partial U}{\partial q}, \tag{3.15}$$

and the second is

$$p\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = p\frac{\partial \dot{x}}{\partial q} = \frac{\partial T}{\partial q}. \tag{3.16}$$

The first equality used Eq. (2.6), and the second, $p = \partial T/\partial \dot{x}$ together with the chain rule.

Gathering the pieces together,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = -\frac{\partial U}{\partial q} + \frac{\partial T}{\partial q}. \tag{3.17}$$

The guess, Eq. (3.11), didn't quite work out, but it's very close, so tinker a bit more. You could put a U on the left hand side to get $T - U$ on both sides. Since $U(x(q,t))$ does not depend upon $\dot{q}$, $\partial U/\partial \dot{q} = 0$. So throw one in and rewrite the previous equation as

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}}\right) - \frac{\partial (T-V)}{\partial q} = 0. \tag{3.18}$$

That's as good as a result as the original guess. Clearly the difference of kinetic and potential energy is important here, so it's given a name.

$$L = T - U. \tag{3.19}$$

is called the **Lagrangian**, and the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{3.20}$$

is the Lagrangian form of the equation of motion.

2.3.2 many degrees of freedom

The many degree-of-freedom case remains to be worked out. Then we have q^i and $\dot{q}^i$ instead of just one q and one $\dot{q}$. It's an exercise in the chain rule, and I'll not provide details. For instance, Eq. (3.13) becomes

$$\frac{\partial T}{\partial \dot{q}^i} = \sum_j p^j \frac{\partial x^j}{\partial q^i}. \tag{3.21}$$

The Lagrangian equations of motion for many degrees of freedom is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \tag{3.22}$$

with the Lagrangian $L = T - U$ as before.

Exercise Decorate the computations of the previous subsection with indices in order to derive equation (3.22).

The coordinate indices can sometimes be a distraction and it is sometimes a good idea to work a derivation through with them left out and then go back to put them in.

Sums over indices which occur twice, such as the j here occur frequently in many contexts. There is a common notational shorthand which consists merely of not writing the summation symbol. For instance,

$$\frac{\partial T}{\partial \dot{q}^i} = p^j \frac{\partial x^j}{\partial q^i}. \tag{3.23}$$

This is called the **Einstein summation convention** – his big contribution to mathematics!. The rule for this convention is that, unless otherwise indicated, repeated indices are summed over $1, \ldots, N$, where N is the number of degrees of freedom.

2.3.3 generalized forces

In the derivation of the Lagrangian equations of motion we have assumed that the forces were derived from a potential. The Lagrangian approach is at its best when the forces are conservative. But, it's worthwhile examining the more general case, so that is done here. We go back to the one-dimensional case because the indices would just be a distraction. Picking up the left hand side of Eq. (3.11) again,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\frac{\partial x}{\partial q}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)\frac{\partial x}{\partial q} + \frac{\partial T}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial q} = F\frac{\partial x}{\partial q} + \frac{\partial T}{\partial q}, \tag{3.24}$$

where F is the ordinary force, which won't be written in terms of a potential this time.

Question How do you legitimize the computation $\partial T/\partial \dot{x}(\partial \dot{x}/\partial q) = \partial T/\partial q$? Isn't something missing? Why is this alright?

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = F\frac{\partial x}{\partial q}. \tag{3.25}$$

The right hand side of this equation is called a **generalized force**. Putting indices back in, as appropriate for a many degree-of-freedom case, the generalized forces are

$$\mathcal{F}^j = \sum_i F_i \frac{\partial x^i}{\partial q^j}. \tag{3.26}$$

Written in terms of generalized forces instead of the potential, the Lagrangian equations take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial q} = \mathcal{F}. \tag{3.27}$$

While we're generalizing everything...the quantity

$$\mathsf{p}^j = \frac{\partial L}{\partial \dot{q}^j} \tag{3.28}$$

is called a **generalized momentum**.

Question Why?

To help avoid confusion with the regular sort of momenta, I will use a sans-serif font for these.

2.4 Examples

Here are some examples illustrating how equations of motion are found in the Lagrangian approach. The general procedure is (1) choose coordinates, (2) compute kinetic and potential energies in some inertial reference frame but expressed in terms of the chosen coordinates, (3) put T and V together into the Lagrangian, (4) compute the derivatives of L, and (5) put them together.

Example 0: the free particle

The kinetic and potential energies of a free particle are

$$\begin{aligned} T &= \frac{1}{2}m|\dot{\mathbf{r}}|^2 \\ V &= 0. \end{aligned} \tag{4.29}$$

The coordinates are chosen as $x^1 = x$, $x^2 = y$ and $x^3 = z$, the usual cartesian coordinates. Then,

$$\frac{\partial T}{\partial \dot{x}^i} = m\dot{x}^i,$$

and the Lagrangian equation of motion is

$$\frac{d}{dt}\left(m\dot{x}^i\right) = m\ddot{x}^i = 0.$$

Well, I hope that should work!

Exercise Work out the equations of motion for a free particle in the plane using polar coordinates.

Example 1: block on a sliding inclined plane.

The first example is a block sliding without friction on an inclined plane, as in Figure 2.3. That's too easy, so there's a slight twist. The inclined plane is not nailed down and it too slides without friction on the horizontal floor. Let us try to compute, for instance, the time it takes the block to slide down the incline.

We'll work it out using the old techniques first. Denote the acceleration of the incline by $\mathbf{A}$ and that of the small block with respect to the incline by $\mathbf{a}$. Breaking them up into components,

$$\begin{aligned} \mathbf{A} &= \ddot{X}\hat{\mathbf{x}} \\ \mathbf{a} &= \ddot{d}(\hat{\mathbf{x}}\cos\alpha - \hat{\mathbf{z}}\sin\alpha). \end{aligned} \tag{4.30}$$

The forces acting on the small block are its weight, and a normal force $\mathbf{F}_N$ from the incline. Since the incline can only move horizontally, the only relevant force is the $\hat{\mathbf{x}}$ component of the reaction force from the small block, so that Newton's 2nd Law gives

$$\begin{aligned} \mathbf{F}_N - mg\hat{\mathbf{z}} &= m(\mathbf{A}+\mathbf{a}) \\ -F_N\sin\alpha &= M\ddot{X}. \end{aligned} \tag{4.31}$$

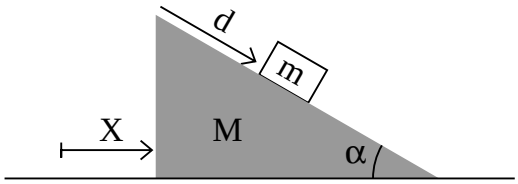

Figure 2.3: A block slides frictionlessly atop an inclined plane, which in turn slides frictionlessly on the floor. X is the horizontal distance of the inclined plane from some fixed point on the floor and d is the distance the smaller block has slid down the incline.

Finally, the normal force is decomposed into components as

$$\mathbf{F}_N = (\hat{\mathbf{x}} \sin\alpha + \hat{\mathbf{z}} \cos\alpha) F_N. \tag{4.32}$$

By substituting the vector decompositions of Eqs. (4.30, 4.32) into the equations (4.31) you obtain three linear equations for three unknowns $\ddot{X}$, $\ddot{d}$ and F_N which you can solve first for one thing and then another to get

$$\ddot{X} = g \frac{\cos\alpha \sin\alpha}{\sin^2\alpha + M/m}, \tag{4.33}$$

and then

$$\ddot{d} = \left(\frac{1 + M/m}{\sin^2\alpha + M/m} \right) g \sin\alpha. \tag{4.34}$$

With $\ddot{d}$ expressed as a constant, it is easy to solve the problem actually posed, since

$$d(t) = d(0) + \dot{d}(0)t + \frac{1}{2}\ddot{d}t^2.$$

So you could do it that way. Now we'll try the Lagrangian method. The first thing is to choose coordinates, which pretty clearly should be X and d. The kinetic energy of the system is the sum of kinetic energies of the two pieces,

$$T = \frac{M}{2}\dot{X}^2 + \frac{m}{2}\left[(\dot{X} + \dot{d}\cos\alpha)^2 + \dot{d}^2 \sin^2\alpha \right], \tag{4.35}$$

and the potential energy is simply the gravitational potential of the small block,

$$V = -mg \sin\alpha \, d. \tag{4.36}$$

Getting the Lagrangian equations of motion is now a fairly mechanical[2] exercise in differentiation. For X,

$$M\ddot{X} + m(\ddot{X} + \ddot{d}\cos\alpha) = 0, \tag{4.37}$$

and for d,

$$m\left[(\ddot{X} + \ddot{d}\cos\alpha)\cos\alpha + \ddot{d}\sin^2\alpha\right] - mg\sin\alpha = 0. \tag{4.38}$$

A slight rearrangement leads to

$$\ddot{X} = -\left(\frac{\cos\alpha}{1 + M/m}\right)\ddot{d},$$

and

$$\ddot{d} = \left(\frac{1 + M/m}{\sin^2\alpha + M/m}\right) g\,\sin\alpha.$$

By doing the computations this way, there was no need ever to have to worry about the force $\mathbf{F}_N$, and there was also less need to break things into components.

Exercise Examine the behavior of the equations of motion as the inclined plane becomes very massive. Does it make sense?

Example 2: double pendulum.

This second example would be very much harder to handle with your old methods. Try it. The system consists of a pair of pendula of lengths ℓ_1 and ℓ_2 with bobs of mass m_1 and m_2 and joined as shown in Figure 2.4.

The potential energy of this system is the sum of that for the two bobs, and can be written down almost immediately,

$$V = -m_1\ell_1\cos\alpha - m_2(\ell_1\cos\alpha + \ell_2\cos\beta). \tag{4.39}$$

The kinetic energy of m_2 is a little more difficult. The result is

$$T = \frac{1}{2}m_1\ell_1^2\dot{\alpha}^2 + \frac{1}{2}m_2\left[\ell_1^2\dot{\alpha}^2 + \ell_2^2\dot{\beta}^2 + 2\ell_1\ell_2\dot{\alpha}\dot{\beta}\cos(\alpha+\beta)\right]. \tag{4.40}$$

To get this, you probably need to write the x and y coordinates of m_2 in terms of the angles as

$$\begin{aligned} x_2 &= \ell_1\sin\alpha + \ell_2\sin\beta, \\ y_2 &= \ell_1\cos\alpha + \ell_2\cos\beta. \end{aligned} \tag{4.41}$$

Once the kinetic and potential energies have been determined, getting the equations of motion is very easy. The equation for α, obtained by differentiating $L = T - V$ with respect to α and $\dot{\alpha}$ is

$$\frac{d}{dt}\left[M\ell_1^2\dot{\alpha} + m_2\ell_1\ell_2\dot{\beta}\cos(\alpha+\beta)\right] + m_2\ell_1\ell_2\dot{\alpha}\dot{\beta}\sin(\alpha+\beta) + Mg\ell_1\sin\alpha = 0, \tag{4.42}$$

[2] no pun intended

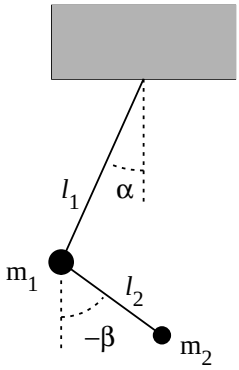

Figure 2.4: A pair of coupled pendula. Both angles are measured clockwise from the vertical, hence the minus sign on β.

while that for β is

$$\frac{d}{dt}\left[m_2\ell_2^2\dot{\beta} + m_2\ell_1\ell_2\dot{\alpha}\cos(\alpha+\beta)\right] + m_2\ell_1\ell_2\dot{\alpha}\dot{\beta}\sin(\alpha+\beta) + m_2 g\ell_2\sin\beta = 0. \tag{4.43}$$

These appear pretty complicated. Fortunately, there is a little cancellation when the time derivates are taken, resulting in

$$M\ell_1^2\ddot{\alpha} + m_2\ell_1\ell_2\ddot{\beta}\cos(\alpha+\beta) - m_2\ell_1\ell_2\dot{\beta}^2\sin(\alpha+\beta) + Mg\ell_1\sin\alpha = 0, \tag{4.44}$$

and

$$m_2\ell_2^2\ddot{\beta} + m_2\ell_1\ell_2\ddot{\alpha}\cos(\alpha+\beta) - m_2\ell_1\ell_2\dot{\alpha}^2\sin(\alpha+\beta) + m_2 g\ell_2\sin\beta = 0. \tag{4.45}$$

Unfortunately, it's still pretty bad. The Lagrangian method allows the differential equations of motion to be determined fairly easily, but it gives no guidance in their solution. Real problems often give complicated equations like these, which are nonlinear. They are very hard to solve without resorting to approximations or numerical techniques. Sometimes, qualitative information about the solutions can be obtained and that is quite valuable in those cases. It's not such an awful thing to have to feed equations of motion to a computer in order to integrate them. But you have to have differential equations first. The machine cannot solve the problem from scratch.

Example 3: bead on a rotating wire.

Here is one more example. This one features a time-dependent constraint. A bead is sliding frictionlessly on a wire which is being rotated by some external agent at a steady rate. The setup is depicted in Figure 2.5. As usual, we want the equations of motion of the bead. The only thing which is not known already is how d changes with time. Choosing that as our coordinate, the kinetic energy is

$$T = \frac{1}{2}m\left[(\omega d\sin\alpha)^2 + \dot{d}^2\right]. \tag{4.46}$$

The potential energy is even easier. It's

$$V = -mgd\cos\alpha. \tag{4.47}$$

Question Why is there a minus sign here?

Forming $L = T - V$ and taking derivatives, the equation of motion pops right out:

$$\ddot{d} - \omega^2\sin^2\alpha\, d + g\sin\alpha = 0. \tag{4.48}$$

This problem has only one degree of freedom. That means that it is completely soluble in a certain sense, as discussed more later. Were you surprised by that, thinking that it looked about as bad as the second example? In fact, since the equation of motion is linear, you can integrate it very easily.

Exercise Integrate the equation of motion. Express $d(t)$ in terms of the initial conditions $d(0)$ and $\dot{d}(0)$. Could that solution break down? When would that happen?

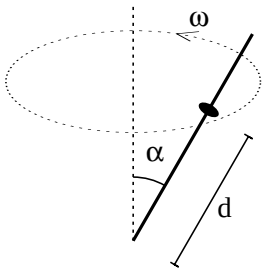

Figure 2.5: The bead slides frictionlessly on wire which is canted at an angle α from the vertical and rotating about the vertical axis at a constant angular velocity ω d is the distance of the bead from the end of the wire.

Example 4: uniformly accelerated frame: an elevator

Suppose you're trapped in an elevator and very bored so you start to do some physics experiments. Naturally you use coordinates referred to the floor. (You can't see out; what else would you use?) The elevator is accelerating upward relative to the earth with acceleration a. Work out the equation of motion for the vertical coordinate of a golf ball tossed into the air.

2.5 Calculus of Variations

The calculus of variations, as it has come to be known, was invented by Jakob and Johann Bernoulli, though Newton had come very close some years earlier. In 1696, Johann Bernoulli [3] solved what is now called the "brachistochrone problem" (see Figure 2.6) and issued a challenge to the other mathematicians of Europe to solve it too. Translated from the Latin:

NEW PROBLEM
WHICH MATHEMATICIANS ARE INVITED TO SOLVE

If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from

[3]The Bernoullis were Swiss. As a result, whether you see the French, German or English versions of their given names is anybody's guess. This is especially confusing since there were so many of them. We have Johann/Jean/John and Jakob/Jacques/Jacob. Fortunately 'Daniel' is invariant.

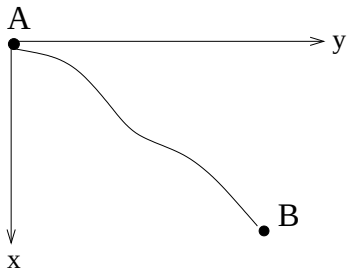

Figure 2.6: The brachistochrone problem: find the curve from A to B that minimizes the time for a particle to slide down under its own weight. The curve shown is not the answer, and neither is a straight line.

> the point A to the point B in the briefest possible time.

Apparently only Leibniz managed to solve it during the six months which elapsed before Bernoulli published his solution:

> PROCLAIMATION
> MADE PUBLIC AT GRÖNINGEN JANUARY 1697
>
> ... six months from the day of publication..., at the end of which, if no one had brought a solution to light, I promised to exhibit my own...

Whereupon, Bernoulli presented his solution and acknowledged Leibniz. Actually, Jakob Bernoulli also solved it, but there was bad blood between the brothers by that time, so I think he 'forgot' to mention it. Isaac Newton heard about the problem from a friend in France, but only late in January 1697. He, like Leibniz, required but a day to polish it off. Although Newton published his solution anonymously, Johann Bernoulli immediately recognized its author and made the remark that one "recognizes a lion from his claw marks." Anyway, I guess that's what people did for fun in those days. It's far too late to answer J. B.'s challenge, but we're going to tackle the problem, too.

The notation is indicated in Fig. 2.6, along with one possible curve going from A to B. Since the particle is falling under its own weight,

$$mgx = \frac{m}{2}v^2.$$

Denoting arc length by s, the total time of descent is

$$T = \int_A^B \frac{ds}{v}.$$

With

$$ds = \frac{ds}{dx}dx = \sqrt{1 + \frac{dy}{dx}^2}\, dx,$$

this becomes

$$T = \int_{x(A)}^{x(B)} \sqrt{\frac{1 + \frac{dy}{dx}^2}{2gx}}\, dx. \tag{5.49}$$

We are not going to carry this particular calculation further at the moment (see MT, §6.3 for more details. I have conformed to their notation.) Instead, let it serve as a prototype of an interesting kind of problem. What can be abstracted here? Given a suggested curve $y(x)$ connecting A to B, which is to say a function of x, the time of descent T is a function of that function, which we could write $T[y]$, with the argument in square brackets as a clue that it is a function itself rather than an ordinary real or complex variable. A thing like this $T[y]$ is sometimes called a **functional**.

The class of all such functions of functions is extremely large, hard to deal with, and mostly useless. Notice that we have a particular kind of functional. It can be expressed as an integral over x of something which depends only upon x, the function y and its first derivative y'. So the brachistochrone problem falls into the class of problems where one has a functional $I[y]$ of a function $y(x)$ which takes the form

$$I[y] = \int_a^b F(y(x), y'(x), x)\, dx, \tag{5.50}$$

for some function $F(\cdot, \cdot, \cdot)$. The function $y(x)$ is allowed to vary over all (suitably smooth) functions with given values $y(a)$ and $y(b)$ at the end points, and the object is to find a function y which extremizes I.

In the brachistochrone problem, with $F(a, b, c) = \sqrt{1 + c^2/2ga}$, we wanted to actually *minimize* T, but we'll generalize that aspect slightly, too. When we hunt for the extrema of a function $y(x)$, we use its derivative. At the extrema, $y' = 0$. Try the same idea with a functional. Suppose I try to sell you $y(x)$, saying that it is an extremum of $I[y]$. Should you buy? Let's see. Suppose it is. We could tweak it a bit by adding a function $\epsilon h(x)$, with $h(a) = h(b) = 0$ so that the values at the end points aren't disturbed and with ϵ a paramater which doesn't depend upon x. (See Figure 2.7.) Then, at the very least, y must extremize $I[\cdot]$ within the small class of functions $y(x) + \epsilon h(x)$ where ϵ is allowed to vary over all real values. Forgetting about all the other ways $y(x)$ could be tweaked, this reduces things to checking whether $\epsilon = 0$ gives an extremum of $I[y + \epsilon h]$. It may not seem to be much, but this is something we can sink our teeth into! For,

$$I[y + \epsilon h] = \int_a^b F(y + h, y' + \epsilon y', x)\, dx, \tag{5.51}$$

and therefore

$$0 = \frac{d}{d\epsilon} I[y + \epsilon h] = \int_a^b \left[h(x) \frac{\partial F}{\partial y} + h'(x) \frac{\partial F}{\partial y'} \right] \Bigg|_{y, y'} dx. \tag{5.52}$$

Now perform an integration by parts on the second term to wind up with

$$\cdots = \int_a^b h(x) \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \Bigg|_{y, y'} dx. \tag{5.53}$$

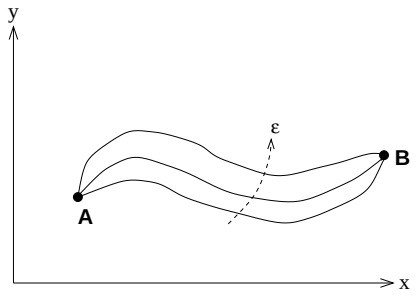

Figure 2.7: A one-parameter family of curves passing from point **A** to **B**. Perhaps the middle curve corresponds to $\epsilon = 0$, the upper to a small positive value and the lower to a small negative value of ϵ.

We were thinking of one particular $h(x)$. But, remember we really wanted Eq. (5.52,5.53) to hold for any h whatever, as long as $h(a) = h(b) = 0$. That gives us so much freedom that the thing multiplying $h(x)$ must be zero:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0. \tag{5.54}$$

This is called the **Euler-Lagrange equation** (for F or for I). Sometimes one says just "Euler equation." It is a necessary condition for y to extremize I. Since any tweaking of f can be described in terms of an h like that above, it seems to be sufficient, too. This is almost the case. (For our purposes, we are as well off as if it were true.)

The reason for "almost" is this. For an ordinary function f, just because $f'(x_0) = 0$, x_0 is not necessarily an extremum. The simplest example is $f(x) = x^3$, which has vanishing derivative at zero. This is a saddle point. A point which is an extremum or a saddle point is called a **stationary point**. The same thing can happen for a functional. The Euler-Lagrange equation is actually an equation for a stationary 'point' of the functional I (where by 'point' I mean a function, really).

We will need the case that the functional I depends upon more that one function, as

$$I[y^1, y^2, \ldots, y^n] = \int_a^b F(y^1(x), y^2(x), \ldots, y^n(x); (y^1)', (y^2)', \ldots, (y^n)'; x)\, dx. \tag{5.55}$$

The Euler-Lagrange equations for this functional are a natural generalization of the one-function equation. They are

$$\frac{\partial F}{\partial y^i} - \frac{d}{dx}\frac{\partial F}{\partial (y^i)'} = 0, \qquad i = 1, 2, \ldots, n. \tag{5.56}$$

This is proved by the observation that freezing in all but one of the y^i's leaves us with the one-function case already considered.

Exercise Finish the brachistochrone problem. Compute the Euler equation, which is a differential equation for the solution. That part's easy. Then solve it. (harder). See Example 6.2 in MT for some help.

2.6 More Calculus of Variations

In my opinion, the calculus of variations might better be called (differential) functional calculus, but the name is stuck now and it's too late. Here's the point of that opinion. In ordinary calculus, we deal with a function, $y(x_1, \ldots, x_n)$ say, which has a finite list of variables. The variables $x_1, \ldots, x_n$ can be thought of as coordinates in a space of n dimensions, and the function f associates a number to each point in this n-dimensional space. The change in $y(\cdot)$ under an infinitesimal shift $(\delta x_1, \ldots, \delta x_n)$ of its argument is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i.$$

You can think of this as a dot product of two n-component vectors $\nabla f \equiv (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and $d\mathbf{x} \equiv (dx_1, \ldots, dx_n)$.

In the variational calculus, we deal with a function of a function, $I[f]$ say. In a certain vague sense, $I[\cdot]$ is a function depending upon an *infinite* number of variables, the values of f at every point x of its domain. Of course, these values are not *quite* independent variables. We can't vary them arbitrarily without destroying the continuity (*a fortiori*, the differentiability) of $y(x)$. But that's basically the idea. So, let's try to find analogs of the ordinary calculus items in this new setting. Changing x_i a bit gives changes f by an amount specified by the partial derivative $\partial f/\partial x_i$. We can try to define a similar object by[4]

$$\frac{\delta I}{\delta y(x)} = \begin{array}{c}\text{proportionality between a small change} \\ \text{in } y(x) \text{ and the change in } I[f]\end{array} \tag{6.57}$$

By comparison to the formula for δf above, we would change the sum to an integral (going to an infinite number of variables) and write

$$\delta I = \int \frac{\delta I}{\delta y(x)} \delta y(x)\, dx. \tag{6.58}$$

Now, looking back to Eqs. (5.51-5.53) in section 2.5, and assuming that I is given by the formula $\int F(y, y', x)\, dx$, you can see that this actually makes sense if we define

$$\frac{\delta I}{\delta y(x)} = \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right\} \Big|_{x, y(x), y'(x)}. \tag{6.59}$$

This is called the **variational derivative**, or the **functional derivative** of $I[y]$.

Now the condition for the function y_0 to be a stationary point of the functional I (quite possibly an extremum, but not necessarily) is written

$$\delta I[y]|_{y_0} = 0.$$

[4]The change of notation from d to δ is traditional and reminds us that we're working in a functional context here.

This is very succinct, and sums it up nicely. The linear change in $I[\cdot]$ under any tiny change of y away from y_0 is zero.

2.7 Hamilton's Principle

So what's the point of all this variational mumbo-jumbo? Be patient just a moment longer and you will see.

Given a starting configuration q^i at time t, and a final configuration $\overline{q}^i$ at time $\overline{t}$, pick any trajectory $q(t)$ which joins them. Define the **Action**

$$S[q] = \int_{q,t}^{\overline{q},\overline{t}} L(q(t), \dot{q}(t), t)\, dt, \tag{7.60}$$

of this trajectory. This is just like the $I[y]$ of the previous two sections, with the replacements of the dependent variable x by t, the function $y(x)$ by $q(t)$, and $F(y, y', x)$ by $L(q, \dot{q}, t)$. The argument q of $S[\cdot]$ refers not to a numerical value, but the entire function $q(t)$.

Now, the stationarity of the action

$$\delta S = 0. \tag{7.61}$$

gives some Euler-Lagrange equations that look like (5.54). Comparing to the Lagrangian form of the equations of motion in section 2.3, you will see that they are precisely the same!

This way of expressing the laws of mechanics given by Eq. (7.61) was discovered by Sir William Rowan Hamilton. Naturally, it is called **Hamilton's Principle**. For some reason, though, he was much more excited over quaternions. Go figure.

Exercise Express the content of Hamilton's Principle in words.

To avoid confusion, it is helpful to establish some terminology for things like the function $q(t)$. An arbitrary path through configuration space or phase space which is parametrized by time will be called a **trajectory**, with the modifier indicating which space if needed. A trajectory is a conceivable history of the system in the event that the laws of physics are repealed. But only some of these trajectories are actually consistent with the dynamical laws. These are called **dynamical trajectories**. These are the ones which could actually happen. Finally, if somebody plotted out a dynamical trajectory for you but forgot to label the points with the appropriate times you would have an **orbit**. This is where the system was in configuration (or phase) space, forgetting about when it was there.

Question One doesn't usually talk about orbits in phase space. If you had one, you could reconstruct the trajectory. How?

2.8 Constraints and Lagrange Multipliers

2.8.1 classifying constraints

One of the very nice things about the Langrangian approach is the way it handles **constraints** and **constraint forces**. For example, the bob of a rigid pendulum free to swing in a plane would have three degrees of freedom in the absence of the rod joining it to the

pivot. But the presence of the rod reduces the number of degrees of freedom to just one. This is a constraint. If the pendulum is allowed to swing freely, there is a force from the rod which ensures the bob follows a circular arc. Since this force ensures the constraint, it is called a constraint force. If we turned the pendulum on its side and spun it around in a horizontal plane, there would also be a constraint force perpendicular to the plane of motion, countering the weight of the bob. In either of these cases, we can avoid dealing with the constraint and the associated force explicitly by choosing coordinates adapted to it. The motion is constrained to be along a circle, so we choose an angular coordinate. That is the only degree of freedom. Similarly, for a marble rolling in a hemispherical bowl, we would use spherical coordinates θ and ϕ.

Life is simplified by choosing coordinates adapted to the constraints because it is not necessary ever to determine the forces of constraint in order to find and solve the equations of motion. However, there are times when one definitely wants to know the constraint forces. It can by a crucially important engineering question whether the structure which is supposed to provide them is actually up to the task. Other times, one simply prefers not to use adapted coordinates.

Or cannot. Some constraints are of a form that it's not even possible to choose coordinates adapted to them. All the constraints mentioned so far have been on the positions only. This sort of constraint can vary with time or not. For the bead on the wire in section 2.4, the constraint is on the position of the bead alone (it's on the wire!), but where those allowed positions are changes with time. All these kinds of constraints can be expressed as

$$\text{some function of position and time} = 0,$$

or equivalently,

$$\text{for each time t, there is a function } G_t(q) \text{ such that } G_t(q) = 0.$$

Question Why are there zeros on the right hand sides? Wouldn't it be more general to allow an arbitrary value? (No.)

These sorts of constraints are called **holonomic**.[5] For holonomic constraints, an adapted system of generalized coordinates can always be found, though the relation between the generalized coordinates and inertial coordinates may be time-dependent.

Question How do you find such coordinates?

There are other kinds of constraints, which are called **nonholonomic**, naturally enough. The standard example of such is a disk which rolls without slipping on a plane. Put a mark on the rim. The configuration of the disk is specified by (a) the point $\mathbf{r}$ at which it touches the plane, (b) the angle ϕ between that point of contact and the mark, and (c) the orientation of the disk on the plane, given for instance by the angle θ between the x-axis and the direction the disk is 'aimed.'

Question How many degrees of freedom are here?

The condition of no slipping means that the disk moves in the direction in which it is aimed, and its rotation rate is tied to its progress in the plane. Explicitly, writing R for the radius of the disk,

$$\begin{aligned} \dot{x} &= R\dot{\phi}\cos\theta, \\ \dot{y} &= R\dot{\phi}\sin\theta, \end{aligned} \tag{8.62}$$

[5]This comes from a greek word meaning "would you like fries with that?" Or maybe I translated wrong.

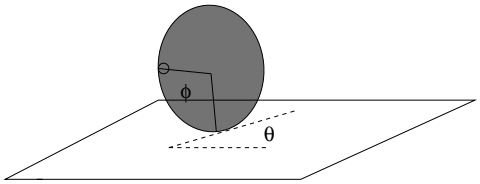

Figure 2.8: A disk rolling without slipping on the plane.

and

$$\text{no restriction on } \dot{\theta}.$$

This constraint places no restiction on the accessible points of the configuration space. If you don't believe it, roll a nickel around on a table for a while and see for yourself. But it does restrict the generalized velocities.

Question Do you want to change your answer to the previous question just a little?

No clever coordinate picking can handle this constraint. The method of Lagrange multipliers will.

2.8.2 Lagrange multipliers

a silly example

Consider the problem of minimizing the function $F(x, y) = x^2 + y^2$ subject to the constraint $y = 2x + 1$. In other words, you want to find the point on the line $y = 2x + 1$ which is closest to the origin. The sensible thing to do is to substitute for y into $F(x, y)$ and then minimize with respect to the single variable x. This is the analog of using coordinates adapted to a constraint.

But suppose we are not sensible. Define $g(x, y) = y - 2x - 1$ so that the constraint is $g = 0$. Now, the change of F under an infinitesimal displacement $\delta\mathbf{r}$ is $\nabla F \cdot \delta\mathbf{r}$. Consequently, F will be minimized along the line $g = 0$ when ∇F is perpendicular to that line. But ∇g is also perpendicular to the line $g = 0$ since δg is certainly zero for any displament along it. Thus, the minimization of $F(x, y)$ subject to the constraint $g(x, y) = 0$ is achieved by finding a point satisfying the constraint at which $\nabla F \propto \nabla g$. Calling the proportionality constant λ,

$$0 = \nabla(F - \lambda g) = (2x + 2y)\hat{\mathbf{x}} + (2y - \lambda)\hat{\mathbf{y}}. \tag{8.63}$$

This is solved by the pair of conditions

$$\lambda = 2y = -x. \tag{8.64}$$

Substituting these into the constraint $g(x, y) = 0$ gives

$$\frac{\lambda}{2} - 2(-\lambda) - 1 = 0 \Rightarrow \lambda = \frac{2}{5}.$$

Thus, the minimum is at $(-2/5, 1/5)$.

Here's the lesson:

> Finding a stationary point of F subject to a constraint $g = 0$ is equivalent to finding a stationary point of $F - \lambda g$, where λ is an initially unknown constant.

The constant λ is called a **Lagrange multiplier**.

Now we can be even more stupid and throw in another extraneous variable, z. We're still trying to find the point on that line in the x-y plane which is closest to the origin, hence minimizes $F(x, y, z) = x^2 + y^2 + z^2$. There is a new constraint. $g_2(x, y, z) \equiv z$ is to be set to zero. Pick any point (x, y) on the line $g = 0$, $g_2 = 0$. Even though it's just a line, we'll call this the **constraint surface**, because that's the general terminology. What is ∇F at that point? It may (and does) have a component along ∇g or ∇g_2. One of these points out of the x-y plane, and the other points away from the line, but staying in the plane. However you slice it, it should be clear that any vector whatever can be decomposed into some combination of these two, plus a component along the constraint surface. At the point which is the answer to the problem, ∇F has no component along the line, so that $\nabla F = \lambda \nabla g + \lambda_2 \nabla g_2$ at that point. Conversely, if $\nabla F = \lambda \nabla g + \lambda_2 \nabla g_2$ at some point on the line, it minimizes F on the constraint surface because g and g_2 have no gradient along it. So the minimum of F subject to the constraints is located by

$$\nabla \{F - \lambda g - \lambda_2 g_2\} = 0.$$

Now we can generalize to many constraints and variables. The function $F(x^1, \ldots, x^n)$ is extremized on the surface defined by the constraints

$$g_1(x^1, \ldots, x^n) = 0, \ldots, g_m(x^1, \ldots, x^n) = 0, \tag{8.65}$$

when

$$\nabla \left\{ F - \sum_{i=1}^{m} \lambda_i \nabla g_i(x^1, \ldots, x^n) \right\} = 0, \tag{8.66}$$

for some collection of numbers $\lambda_1, \ldots, \lambda_m$.

application to mechanics

Many isn't quite enough, however. For dynamics problems we need an infinite number of variables and constraints. Remember that, roughly speaking, the variables in a mechanics problem are all the positions at each instant of time over some interval. For instance, try to extremize $S = \int_0^a L dt$ with the constraint $q^1 = 0$. Since each $q^1(t)$ for each value of t is basically an independent variable, there are actually an infinity of constraints: $q^1(t) = 0$ for each t. Sometimes the constraint really depends on the time in the more ordinary sense and this is clearer. In any event, it is now easy to extrapolate the earlier results to

$$\delta \left\{ S - \int \lambda(t) \delta G_t(q(t), \dot{q}(t))\, dt \right\} = 0. \tag{8.67}$$

Expressing the action S in terms of a Lagrangian, the expression to be made stationary is $\int (L - \lambda G)\, dt$. This looks just like the ordinary problem without Lagrange multipliers, except that L is replaced by

$$\tilde{L} = L - \lambda G. \tag{8.68}$$

So the amended equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\right) - \frac{\partial \tilde{L}}{\partial q^i} = 0. \tag{8.69}$$

2.9 Symmetries and Conservation Laws

Imagine a solar system, far away from anything else. The planets move in their appointed courses about the star. Freeze in the positions and velocities, transport the whole thing a million miles away and restart it. What happens? Nothing really. The planets move along exactly the same orbits as before, just shifted by a million miles. Now imagine shifting just the planets, but not the star. This time, the motions after the shift look very different from before. The first shift, which moves all the bodies, is a symmetry of the dynamics, and the second is not. This may seem exceedingly pointless, but there is a deep connection between symmetries and conservation laws, as we'll see in a bit. Results of the symmetry $\Rightarrow$ conservation law sort arise in slightly different form in many fields, and are collectively referred to as Noether's theorem, after the mathematician Emmy Noether.

Quite generally, a symmetry is an invariance of something under a transformation. Most familiar are symmetries of form. The form of a square is symmetric under rotation by 90° about its center. A circle is symmetric under any rotation about its center. Here we are concerned with symmetries of the Lagrangian. Let's see what that means. The Lagrangian is a function of positions and velocities, so a symmetry of the Lagrangian is a transformation on the positions and velocities which does not alter the value of the Lagrangian:

$$L(\phi(q), \phi(\dot{q}), t) = L(q, \dot{q}, t).$$

The example was a symmetry of the Lagrangian. The kinetic energies of the planets and star depends only upon their velocities. The potential energy depends only upon their relative positions. So, as long as the whole system is moved as a unit, both the kinetic and potential energies are unaltered, and so is the Lagrangian. You can see from this example that the transformation on the velocities is somehow already implicit in the change of the coordinates. We won't nail that connection down precisely since it won't be needed.

Symmetries of the Lagrangian carry dynamical trajectories onto dynamical trajectories. This could be established from the Lagrangian equations of motion, but it is one of those questions better approached with Hamilton's Principle. The symmetry preserves actions too, after all.

$$\int_a^b L(q, \dot{q}, t)\, dt = \int_a^b L(\phi(q), \phi(\dot{q}), t)\, dt.$$

As a result, if the action is stationary for $t \mapsto q(t)$, it is also stationary for $t \mapsto \phi(q(t))$ because a small variation of $q(t)$ corresponds to a small variation of $\phi(q(t))$ by means of ϕ and vice versa.

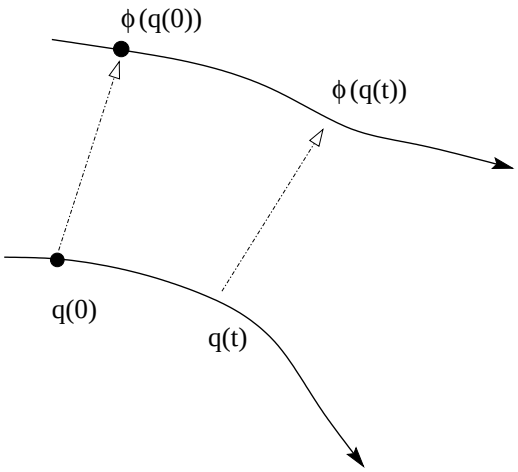

Figure 2.9: A transformation on configuration space induces a transformation on trajectories and on velocities.

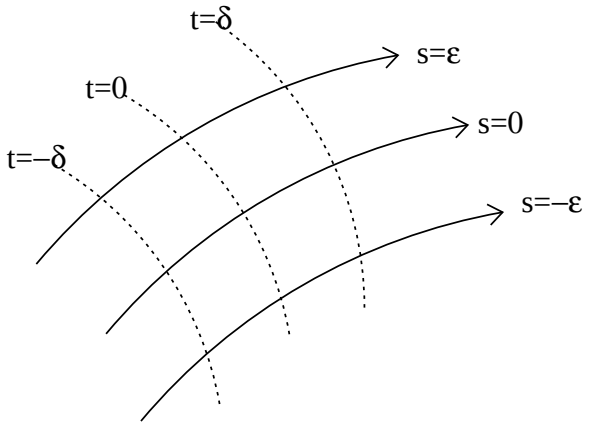

Figure 2.10: A family of dynamical trajectories generated by applying a continuous symmetry. The solid lines are the trajectories and the dashed lines cut across at particular values of the time.

That's a nice thing to know, but its only a particular kind of symmetry which corresponds to a conservation law. What is needed is a **continuous symmetry**. Translations of an isolated system are a good simple example. The amount by which we translate is at our disposal and can be made arbitrarily small. The symmetry is **parametrized** by **a**, which is precisely the vector by which we translate, and which is a continuous variable. Restricting attention to the translations along a particular direction, there is one parameter, conveniently chosen as the distance.

2.9.1 cyclic coordinates and first integrals

The simplest situation is a Lagrangian which does not depend on one of the generalized coordinates, say q^i. That is,

$$\frac{\partial L}{\partial q^i} = 0. \tag{9.70}$$

A coordinate of this sort is called **cyclic**. In that case, the Lagrangian equation of motion for q^i is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0. \tag{9.71}$$

In other words, the generalized momentum

$$\frac{\partial L}{\partial \dot{q}^i},$$

does not change with time.

For a single free particle, the Lagrangian is invariant under any shift of its position. This tells us that the ordinary momentum is conserved. When the Lagrangian is invariant under rotation about a given axis, the associated momentum which is conserved is an angular momentum. A smooth function of q and $\dot{q}$ such as this, which is constant along any dynamical trajectory is a conserved quantity. Two other common names for such a thing are **first integral**, and **constant of the motion**.

There is a sense in which the whole story of this connection between symmetries and conservation laws is told by cyclic coordinates. However, one may not be so lucky as to have symmetries of the Lagrangian so neatly 'lined up' with the coordinates. For example, with an isolated many-particle system, such as the solar system discussed at the beginning of this section, there is no cyclic coordinate if we take the coordinates of the individual particles as our generalized coordinates. If we split off the center of mass motion, then there are three — the coordinates of the center of mass. In this example it wasn't so hard to hunt down a system of coordinates which made the symmetry obvious. But in general it may not be, or may not be convenient. That's the situation addressed next.

2.9.2 Noether's theorem

So, we consider a general one-parameter continuous symmetry. Call the parameter s, and denote the symmetry operation which goes with that value ϕ_s. Find some dynamical trajectory, and label it with $s = 0$. Now, by applying ϕ_s for values of s very close to zero, obtain

a family of ever-so-slightly different dynamical trajectories. This is depicted in Figure 2.10. Let's change notation a bit. The original trajectory becomes $q(t,0)$ and the one obtained by applying ϕ_s is $q(t,s)$ in such a way that the t values match up, that is, $q(0,0)$ goes into $q(0,s)$ under the action of ϕ_s.

This construction produces a configuration $q(t,s)$ which is a function of two real variables. A couple of things are known about this $q(t,s)$. If the second argument, s, is held constant, the Lagrangian equations of motion hold:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{9.72}$$

What used to be a total derivative with respect to t on the first term is now a partial derivative to indicate that s is held constant. The second thing known is that shifting s is a symmetry:

$$0 = \frac{\partial L}{\partial s} = \sum_i \left(\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial s}\right). \tag{9.73}$$

Also, $\dot{q}^i$, which is the time rate of change of q^i with s fixed, can also be written $\partial q^i/\partial t$, so that

$$\frac{\partial \dot{q}^i}{\partial s} = \frac{\partial^2 q^i}{\partial t \partial s} = \frac{\partial}{\partial t}\frac{\partial q^i}{\partial s}. \tag{9.74}$$

Now everything is set up. Substitute Eq. (9.74) into the last term of Eq. (9.73), integrate it by parts and notice that a piece of the result is precisely the left hand side of the Lagrange equations of motion (Eq. 9.72), hence can be discarded. The first two steps are

$$\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s} = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s}\right] - \frac{\partial q}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right). \tag{9.75}$$

And, after putting this back in,

$$\frac{\partial}{\partial t}\left[\sum_i \frac{\partial L}{\partial \dot{q}^i}\frac{\partial q^i}{\partial s}\right] = 0. \tag{9.76}$$

This is really the final result. It says that the quantity

$$\sum_i \frac{\partial L}{\partial \dot{q}^i}\frac{\partial q^i}{\partial s}$$

is constant along any dynamical trajectory, which is a conservation law. It only remains to identify $\partial q/\partial s$ in any given case.

Here is an important example of the theorem in action. Suppose a mechanical system containing many particles has rotational symmetry about some point. If, for instance, we are dealing with an isolated system interacting by central forces, that will be the case. A continuous symmetry will be rotations by θ about $\hat{\mathbf{z}}$. θ here corresponds to the parameter s in the theorem. You may as well pick any axis to do the rotation about and then call it

z. At any rate, rotation by an infinitesimal angle $d\theta$ has this effect on the positions of the particles:

$$d\mathbf{r}_\alpha = \hat{\mathbf{z}} \times \mathbf{r}_\alpha d\theta. \tag{9.77}$$

In case this doesn't look familiar, we'll come back to it in a bit. Rewriting in Cartesian coordinates,

$$\begin{aligned} dx_\alpha &= -y_\alpha d\theta \\ dy_\alpha &= x_\alpha d\theta. \end{aligned} \tag{9.78}$$

Now, our q's are x's, y's and z's, and s is θ. So,

$$\begin{aligned} \frac{\partial x_\alpha}{\partial \theta} &= -y_\alpha \\ \frac{\partial y_\alpha}{\partial \theta} &= x_\alpha \\ \frac{\partial z_\alpha}{\partial \theta} &= 0. \end{aligned} \tag{9.79}$$

The potential does not depend upon the velocities, leaving

$$\frac{\partial L}{\partial \dot{x}_\alpha} = m_\alpha \dot{x}_\alpha, \tag{9.80}$$

with a very similar equation for y's and z's. Putting it all together,

$$0 = \sum_\alpha m_\alpha (x_\alpha \dot{y}_\alpha - y_\alpha \dot{x}_\alpha) = \sum_\alpha L_\alpha^z = L^z. \tag{9.81}$$

The z component of angular momentum is conserved. This result allows us to view the conservation of angular momentum as essentially a consequence of rotational symmetry of isolated systems.

Exercise Following the example, demonstrate conservation of momentum from translational symmetry. That is, assume the Lagrangian is unchanged when the positions of all particles are shifted by the same constant amount.

Question How does the case of a cyclic coordinate fit into this more general framework?

2.9.3 enter the Hamiltonian

Very often, a Lagrangian does not depend upon time explicitly:

$$\frac{\partial L}{\partial t} = 0.$$

This is a bit like the case of a cyclic coordinate, although time isn't really a coordinate in that sense. Maybe it nonetheless leads to a conservation law? Hmm, let's see:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}. \tag{9.82}$$

If we integrate by parts on the second term we can get a total time derivative, just the sort of thing we're looking for. And the leftover term can be combined with the first:

$$\frac{dL}{dt} = \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right). \tag{9.83}$$

Aha! the square bracket term is zero by the EOM. So,

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right). \tag{9.84}$$

Yes! it *is* a conservation law. This new conserved quantity is important enough to be named after somebody.

$$H = \sum_i \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \tag{9.85}$$

is called the **Hamiltonian**.

Exercise Check that the indices were put back on correctly.

Usually, the Hamiltonian is nothing other than the energy. That is the case if the potential depends only on the configuration coordinates, and the kinetic energy takes the form

$$T = \sum_{ij} T_{ij}(q)\dot{q}^i\dot{q}^j, \tag{9.86}$$

for some collection of functions $T_{ij}(q)$. Most of the time you arrange for T_{ij} to be a constant and zero even if $i \neq j$. However, the kinetic energy generally does not take this form for moving coordinate systems. In that case we get a new conservation law.

example The simplest possible example of a Hamiltonian which is not the energy is given by a free particle moving in one dimension, but described in reference frame uniformly accelerated with acceleration a (think about the elevator). So, x is the generalized coordinate, and

$$x + \frac{1}{2}at^2 \tag{9.87}$$

is the position in an inertial reference frame.

Now, since there is no potential energy, the Lagrangian is equal to the kinetic energy:

$$L = \frac{m}{2}\left(\dot{x} + at\right)^2. \tag{9.88}$$

Computing the Hamiltonian directly according to the recipe,

$$H = \frac{m}{2}\left(\dot{x}^2 + (at)^2\right). \tag{9.89}$$

This is most definitely not the energy!

We have not seen the last of the Hamiltonian. It will be back later, and in a rôle that may surprise you.

2.10 Notes

This chapter combines most of the material from chapters 6 and 7 of MT. There is more on the Hamiltonian formalism in MT chapter 7, but we are putting that off till later.