Chapter 4

From 1-d Motion to the Kepler Problem

4.1 Overview

In the previous chapter we studied a very restricted class of motion with one degree of freedom, which was linearized dynamics around fixed points. There are many situations to which that is applicable as-is. There are others to which it makes a good first approximation. Actually, energy-conserving dynamics with one degree of freedom is simple enough that the simplification afforded by linearization isn't essential to finding solutions. That's the topic of this chapter. The Kepler problem is tied up with this because it's very nearly one degree of freedom. In the next chapter we'll start looking at more degrees of freedom, starting with linearized dynamics again, as we did for one degree of freedom.

Section 4.2 treats the general problem of one-dimensional motion, reducing it to a 'simple' exercise in integration. This is illustrated on one of our favorite examples, the plane pendulum. Then, the problem of two bodies interacting through a central force is taken up in section 4.3. Regardless of the force law involved, this problem can be reduced to two degrees of freedom by exploiting conservation laws. It is equivalent to a problem of a single particle going around a fixed center of force. We'll see that there is generally no reason to expect the orbits to be closed. (See figure 4.7 if that's obscure.) For the special case of an inverse square law, the orbits are closed. Since you are familiar with the fact that planetary orbits are ellipses, you are probably not as surprised by this as you should be. Anyway, in section 4.4, we'll derive Kepler's laws and work out explicit formulas for the orbits in terms of energy and angular momentum.

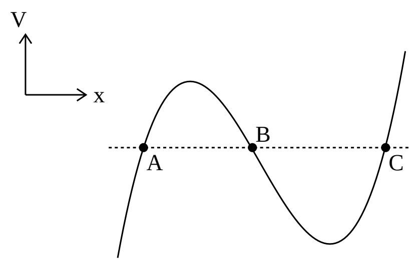

Figure 4.1: Turning points in a potential for one-dimensional motion.

4.2 Motion in One Dimension

In this section, we're going to see how conservative one-dimensional motion is, in a certain sense, a completely soluble problem. 'Conservative' means energy conserving as a general rule. One might better say 'Hamiltonian,' but since that won't make sense to you for a while, we'll forebear. The idea is most cleanly exhibited by thinking about motion of a particle along a line. Other cases require some modification of the formulas, but the ideas are precisely the same.

4.2.1 the general idea

A great deal can be learned about the motion from simply looking at a plot of the potential. An example appears as Figure 4.1. The energy of the particle is indicated by the dashed line. Since the kinetic energy must be positive, the particle can only be in locations where the potential is below that line. The points where the energy level intersects the graph of V are then **turning points**. For the indicated energy, the particle can oscillate back and forth forever between the points B and C.

Question There is another orbit with that same energy, and it has A as a turning point. What does it look like?

This simple analysis of a picture has told us quite a bit for the effort invested, and demonstrates the power of the energy concept. But, something a little more quantitative would be nice. The total energy of our particle is

$$E = T + V = \frac{m}{2}\dot{x}^2 + V(x). \tag{2.1}$$

Question The additional complication which could arise (with a non-Cartesian

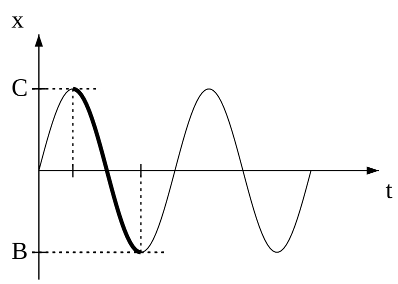

Figure 4.2:

coordinate) is in the kinetic energy term. What can happen?

Now, you can solve the energy equation for $\dot{x}$ to get

$$\frac{dx}{dt} = \pm\sqrt{T} = \pm\sqrt{\frac{2}{m}(E - V(x))}. \tag{2.2}$$

This can be immediately integrated, yielding

$$t = \int_{x(0)}^{x(t)} \frac{|dx'|}{\sqrt{\frac{2}{m}[E - V(x')]}}. \tag{2.3}$$

This yields t as a function of x, kind of a strange thing, but it be inverted to find $x(t)$. By 'immediately' I simply mean that we can write that integral down right away. Rendering a problem into this form of one or more single variable integrals goes by the quaint and musty sounding name of **reduction to quadratures**.[1] This certainly does not mean that we can express the integrals in closed form. They may need to be evaluated numerically, but that is something which can be done quickly and accurately by computer.

Some care is needed in interpreting equation (2.3). Since the motion under consideration is periodic, x is not a single-valued function of t, so the integral cannot be generally valid as it stands. That is not hard to patch up, but there is no need – if you've got $x(t)$ over half a period (period $= \tau$), you've got it for all time. The second half-period is the first in reverse and then $x(t + \tau) = x(t)$. For instance, for the potential in figure 4.1, $x(t)$ might look like figure 4.2, we could get everything we

[1] I think I saw a quadrature once, but that was at night and it flew away quickly, so I'm not really sure.

need from the heavily shaded part of the graph. Integrating over this whole range gives half the period:

$$\frac{\tau}{2} = \int_{\text{turning pt 1}}^{\text{turning pt 2}} \frac{|dx'|}{\sqrt{\frac{2}{m}[E - V(x')]}}. \tag{2.4}$$

Exercise Compute $t(x)$ and then $x(t)$ for the simple harmonic oscillator by the method presented here. Compare to that of the previous chapter.

Question Sometimes people compare one-dimensional motion in a potential $V(x)$ to a particle sliding frictionlessly on a landscape having a height profile looking just like the graph of $V(x)$. What is the difference between these two situations?

4.2.2 phase space view: plane pendulum again

The ordinary plane pendulum is a system with one degree of freedom. It's linearization was discussed toward the beginning of Chapter 3, where there is a plot of the potential energy. Its total energy

$$E = \frac{m\ell^2}{2}\dot{\theta}^2 + mg\ell(1 - \cos\theta), \tag{2.5}$$

has precisely the form discussed a bit earlier, even though θ is not a Cartesian coordinate. As a result, the method presented there can be used to determine the motion. We will come back to the problem of finding the period by that method shortly. First, I want to draw more pictures.

The picture I want to draw is Figure 4.3, which is the phase portrait of the plane pendulum. Recall that precisely one orbit passes through each point of phase space. The phase portrait depicts these orbits, though of course you can't actually draw them all.

To obtain the orbits, we should in principle be obliged to solve the equation of motion,

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta, \tag{2.6}$$

which can be derived from the Lagrangian

$$L = \frac{m\ell^2}{2}\dot{\theta}^2 - mg\ell(1 - \cos\theta). \tag{2.7}$$

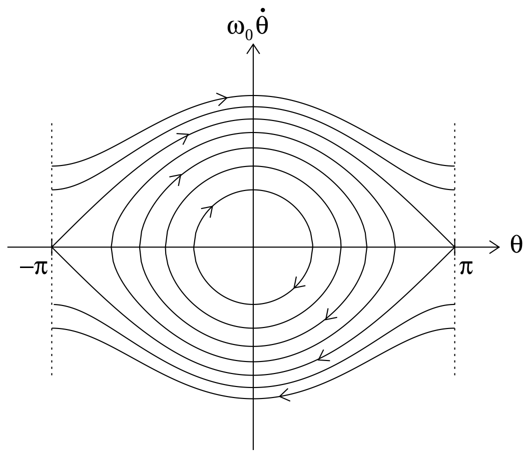

Figure 4.3: Energy isosurfaces in the phase plane of the ordinary pendulum. These are also the orbits.

In order to better express the spirit of the phase space picture, we'll change notation a little. The phase space has coordinates θ and $\dot{\theta}$. The (second order) equation of motion can be rewritten as the pair of first order equations

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} \\ \frac{d\dot{\theta}}{dt} &= -\frac{g}{\ell}\sin\theta. \end{aligned} \tag{2.8}$$

In order to find the orbits, these need to be integrated. But that's not how Figure 4.3 was really made. What's actually shown there are surfaces of constant energy, according to Eq. (2.5). The two are the same for one degree of freedom, and only in that case. In general, the set of points in phase space having a given value of energy is a surface in phase space having dimension one less than the original phase space. This is called an **energy isosurface**, an **energy surface**, or sometimes an **energy shell**. You can think of the phase space as being filled out with nested energy surfaces like the layers of an onion. With one degree of freedom, the energy surfaces are curves. In that case, there is no room on the energy surface for an orbit to wander. So, the two notions must coincide. With more degrees of freedom, specifying just the energy does not tie everything down in that way.

To be quite precise, an energy surface in a one-dimensional phase space can consist of more than one orbit. This occurs for the pendulum in fact. The low energy orbits are oscillatory. Those are the ones looking like circles or ellipses in the figure. The high energy orbits correspond to a pendulum with so much energy that it spins round and round the support. But it can spin around in two different directions, so at these high energies, there are two orbits with the same energy.

Finally, there are three special orbits. Suppose the pendulum is swinging clockwise and goes through $\theta = 0$ at time zero, and that its total energy is equal to the potential energy at $\theta = \pi$. Then, it will approach $\theta = \pi$ in the distant future, but it will never quite make it. If you extrapolate its motion into the past, you find that $\theta \to -\pi$ as $t \to -\infty$. It could also go in the opposite direction. These two orbits are shown on the figure as the curves which just touch $\theta = \pm\pi$, $\dot{\theta} = 0$. Finally, the single point $\theta = \pi, \dot{\theta} = 0$ is an orbit.

Question How can the single point $(\pi, 0)$ be an orbit? Besides, even if it is, isn't there another at $(-\pi, 0)$?

Exercise Draw the phase portrait for the potential in Figure 4.1.

4.2.3 the period of the pendulum

Now come back to the problem of computing the period by the methods of section 4.2.1. The period of a simple harmonic oscillator is independent of its amplitude. This follows directly from linearity of the equation of motion. The period of a real pendulum will start to deviate from its small oscillation period when the amplitude grows large enough for it to realize that the potential is not really harmonic. In figure 4.3 that corresponds to the energy at which the orbits no longer look quite circular.

Question Should the period be shorter or longer for larger amplitudes?

First, we'll rewrite the potential by means of a double angle formula,

$$1 - \cos\theta = 2\sin^2(\theta/2).$$

Then,

$$\frac{m}{2}\ell^2\dot{\theta}^2 = E - mg\ell[2\sin^2(\theta/2)].$$

The frequency of oscillation for small vibrations of the pendulum is

$$\omega_0^2 = \frac{g}{\ell}. \tag{2.9}$$

Let's clean things up a bit by using the inverse of this as our unit of time, so that a new dimensionless time variable is $\bar{t}$:

$$\bar{t} \overset{def}{=} \omega_0 t \Rightarrow \frac{d\theta}{d\bar{t}} = \omega_0^{-1}\dot{\theta},$$

and a dimensionless energy is

$$\overline{E} \stackrel{def}{=} \frac{E}{mg\ell}.$$

This gives (conservation of energy)

$$\frac{d\theta}{d\overline{t}} = \pm\sqrt{2\left[\overline{E} - 2\sin^2\left(\frac{\theta}{2}\right)\right]}. \tag{2.10}$$

Finally, the pendulum's turning point θ_0 corresponds to $\dot{\theta} = 0$ (all the energy is potential at that point!):

$$2\sin^2\frac{\theta_0}{2} = \overline{E} = \frac{E}{mg\ell}. \tag{2.11}$$

In one quarter-period, the pendulum swings from $\theta = 0$ to $\theta = \theta_0$, so with $\overline{\tau} = \omega_0\tau$,

$$\frac{\overline{\tau}}{4} = \int_0^{\theta_0} \frac{d\overline{t}}{d\theta}\, d\theta = \int_0^{\theta_0} \frac{d\theta}{2\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}.$$

To handle this dreadful looking thing, the clever change of variables

$$\sin\beta = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \tag{2.12}$$

comes in very handy. Since the right hand side ranges from -1 to 1, this makes sense. That reduces the quarter-period equation to

$$\frac{\overline{\tau}}{4} = \int_0^{\theta_0} \frac{d\theta}{2\sin(\theta_0/2)\sqrt{1-\sin^2\beta}}. \tag{2.13}$$

Of course we've no intention of leaving it like this. A couple of side computations are needed to make substitutions. Differentiating the change of variable equation,

$$\frac{d\theta}{2}\cos\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta_0}{2}\right)\cos\beta\, d\beta$$

Also from that formula,

$$\cos(\theta/2) = \sqrt{1 - \sin^2(\theta_0/2)\sin^2\beta}.$$

Combining these last two,

$$\frac{d\theta}{2\sin(\theta_0/2)} = \left[1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\beta\right]^{-1/2}\cos\beta\, d\beta.$$

Now, stuffing it all into equation (2.13),

$$\frac{\overline{\tau}}{4} = \int_0^{\pi/2} \left[1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\beta\right]^{-1/2} d\beta. \tag{2.14}$$

At first glance we don't seem to be much better off, but this allows a systematic expansion in $\sin^2(\theta_0/2)$. This is a sensible expansion parameter because it goes to zero with the amplitude. Thus,

$$\frac{\overline{\tau}}{4} = \int_0^{\pi/2} \left[1 + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\sin^2\beta + \cdots\right] d\beta. \tag{2.15}$$

The nice thing about this expression is that the integrals this entails are all very easy, since they are just powers of $\sin(\theta/2)$, and you can churn out terms till the cows come home. Let's just do one. The result is

$$\frac{\overline{\tau}}{2\pi} \approx 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}. \tag{2.16}$$

Recall that in the units we're using, the small oscillation period is 2π. This first correction indicates an increase of the period at higher amplitudes. Was your answer to the Question correct?

Exercise Work out the two terms in equation (2.16).

4.3 Two Particles Moving in a Central Force Field

A force exerted by one body on another which acts along the vector joining the two is called a **central force**. According to the discussion in section 1.6, a central force can be derived from a potential $V(r)$ depending upon only the distance r between the two bodies. In this section we will consider the general problem of two particles interacting via a central force. The very important special case of an inverse square force is taken up in the next section. Each of the particles has three degrees of freedom, making a total of six, so the phase space is twelve dimensional. We have seen how conservation of energy can be used to find subspaces on which the motion takes place, reducing problems with one degree of freedom to child's play. Much more cutting down is needed before a tractable problem is obtained, so more constants of the motion will have to be exploited. The idea is the same though. Each constant of the motion reduces by one the dimension of the space on which the motion takes place.

4.3.1 Reduction

The Langrangian for a pair of particles with masses m_1 and m_2 interacting through a central force is then

$$L = \frac{m_1}{2}|\dot{\mathbf{r}}_1|^2 + \frac{m_1}{2}|\dot{\mathbf{r}}_2|^2 - V(r). \tag{3.17}$$

The pair of particles is a six degree-of-freedom system, so that the phase space has twelve dimensions. Fortunately, things can be cut down considerably. By systematically exploiting conservation principles, we can reduce the number of "effective" degrees of freedom to two, regardless of the form of $V(r)$. For the special case of an inverse-square law, the problem magically reduces even further, to only one degree-of-freedom.

(1) Conservation of P

The first conservation principle we'll use is that of total momentum $\mathbf{P}$, which follows from translation invariance of the Lagrangian. To exploit this, we split off the center of mass motion. Write

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r} \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \end{aligned} \tag{3.18}$$

Then, substitute into Eq. (3.17), do some simplification, and wind up with the Lagrangian expressed as

$$L = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{\mu}{2}\dot{\mathbf{r}}^2 - V(r), \tag{3.19}$$

with the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Now $\mathbf{R}$ is a cyclic coordinate (or three), so it's equation of motion reduces to

$$\frac{d\mathbf{P}}{dt} = 0,$$

where $\mathbf{P}$ is the total momentum of the system. Since the center of mass motion has been split off and is doing its own thing (which is not very much), We have reduced the problem to the three degrees of freedom comprising $\mathbf{r}$.

(2) Conservation of L

Total angular momentum is also conserved. Via Noether's theorem, this conservation law is related to invariance of the Lagrangian under identical rotations of both $\mathbf{r}_1$ and $\mathbf{r}_2$, or equivalently of $\mathbf{R}$ and $\mathbf{r}$. The angular momentum for this system is

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mu \mathbf{r} \times \dot{\mathbf{r}}. \tag{3.20}$$

In fact, there is a bigger invariance. The Lagrangian is invariant under rotating $\mathbf{R}$ alone, or $\mathbf{r}$ alone. This means that

$$\mathbf{L}_{\mathrm{CM}} = \mathbf{R} \times \mathbf{P}$$

and

$$\mathbf{L}_{\mathrm{internal}} = \mu \mathbf{r} \times \dot{\mathbf{r}}$$

are separately conserved. This conclusion can be reached by more pedestrian arguments, as follows:

$$\frac{d\mathbf{L}_{\mathrm{CM}}}{dt} = \dot{\mathbf{R}} \times \mathbf{P} + \mathbf{R} \times \dot{\mathbf{P}},$$

and each term on the right hand side is zero (but for different reasons). So, the first term on the right hand side of equation (3.20) is constant. Since we know the left hand side is also constant, the remaining term must be as well.

In fact, we could have avoided some of this argumentation by simply deciding to work in the center of mass frame. Then $\mathbf{R} = \mathbf{P} = 0$ and we simply drop those terms. We'll do that now and write

$$L = \frac{\mu}{2}\dot{\mathbf{r}}^2 - V(r). \tag{3.21}$$

Since

$$\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}}$$

is constant, $\mathbf{r}$ and $\dot{\mathbf{r}}$ define some plane which is not changing with time (see figure 4.4). Choose coordinates so that $\mathbf{L}$ is oriented along $\hat{\mathbf{z}}$ and use polar coordinates in the x-y plane, so that $\mathbf{r} = (r\cos\theta)\hat{\mathbf{e}}_x + (r\sin\theta)\hat{\mathbf{e}}_y$. This reduces the Lagrangian to

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{3.22}$$

How many degrees of freedom are we down to? Two: r and θ. But the motion is confined to a three dimensional subspace of phase space. By choosing coordinates as we have, $\mathbf{L}$ is known to be in the $\hat{\mathbf{e}}_z$ direction, so just write

$$\mathbf{L} = \ell\hat{\mathbf{e}}_z. \tag{3.23}$$

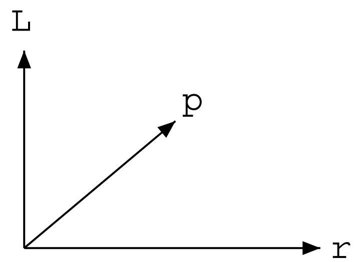

Figure 4.4: The separation vector and momentum in the center of mass frame define an invariant plane.

Then ℓ is the generalized momentum associated with θ:

$$\ell = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}. \tag{3.24}$$

The constancy of ℓ involves one coordinate (r) and one generalized velocity $(\dot{\theta})$. Thus it does not eliminate a degree of freedom, but there are surfaces in the four-dimensional $(r, \theta, \dot{r}, \dot{\theta})$ phase space to which the orbits are confined. This is similar to what happened with energy conservation for the plane pendulum.

(3) Conservation of E

Speaking of energy, that is the final conserved quantity we can exploit. This conservation law is associated with the fact that the Lagrangian does not have any explicit dependence on the time. The kinetic energy of the two-particle system is a sum of a term arising from center-of-mass motion and a term from the relative motion. Since we are working in the center of mass frame, the first of these has been eliminated, leaving

$$E = \frac{\mu}{2}|\dot{\mathbf{r}}|^2 + V(r) = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r). \tag{3.25}$$

It turns out to be convenient to combine these last two terms by defining

$$V_{\text{eff}}(r) \stackrel{def}{=} \frac{\ell^2}{2\mu r^2} + V(r). \tag{3.26}$$

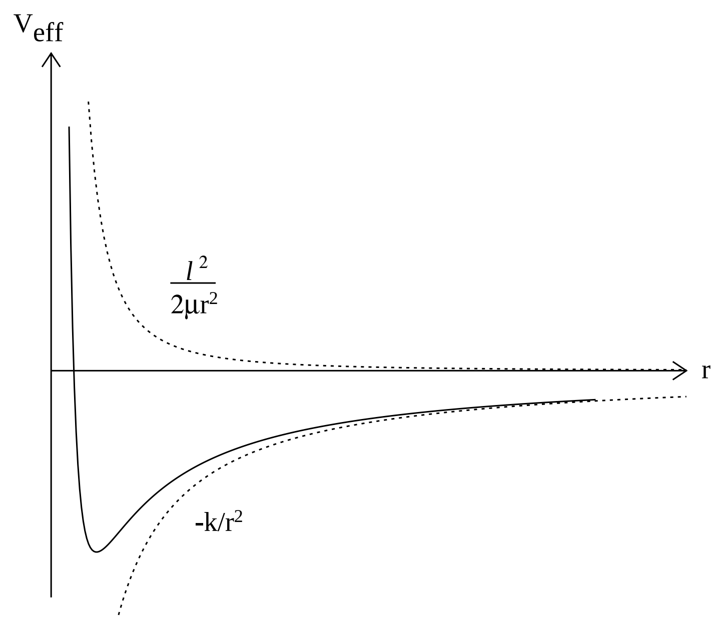

Figure 4.5: The effective potential (solid curve) for the radial motion and the pieces of which it is composed. The divergence in V_{eff} at small r due to the angular momentum term is called a **centrifugal barrier**

The point of this is that the motion looks like that of a particle moving in one cartesian dimension subject to precisely this potential. In fact,

$$E = \frac{\mu}{2}\dot{r}^2 + V_{\text{eff}}(r), \tag{3.27}$$

and solving for $\dot{r}$,

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}(E - V_{\text{eff}})}}. \tag{3.28}$$

Also, from Eq. (3.22), the equation of motion for r is

$$\mu\ddot{r} + \frac{dV_{\text{eff}}}{dr} = 0. \tag{3.29}$$

So it really does look like what's in section 4.2.1. Once equation (3.28) has been solved for $r(t)$, conservation of $\mathbf{L}$ allows $d\theta/dt$ to be found:

$$\dot{\theta(t)} = \frac{\ell}{\mu r(t)^2}.$$

Alternatively, this equation can be used to elimintate t from Eq. (3.28), resulting in an equation for the orbit $r(\theta)$,

$$\frac{dr}{d\theta} = \frac{r^2}{\ell}\sqrt{2\mu(E - V_{\text{eff}}(r))}, \tag{3.30}$$

or

$$\theta(r_1) - \theta(r_2) = \int_{r_1}^{r_2} \frac{\ell/r'^2\, dr}{\sqrt{2\mu(E - V_{\text{eff}}(r'))}}. \tag{3.31}$$

The effective potential for the radial motion confines the orbit to the annular region between r_{min} and r_{max} as depicted in Figure 4.6. r bounces back and forth in a perfectly periodic fashion. Meanwhile, θ advances at a rate which varies with r. Points at which the orbit reaches r_{min} are **apocenters** and those at which it reaches r_{max} are **pericenters**. If you know the greek name for whatever the body is orbiting, the socially correct thing to do is to replace 'center' by that, e.g. for the sun, 'apohelion' and 'perihelion', for the earth, 'apogee' and 'perigee.'

At any rate, a half cycle from apocenter to pericenter is enough to tell the whole story (just as in section 4.2.1). Between those two points, θ advances by an angle

$$\Phi = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\ell/r^2\, dr}{\sqrt{2\mu(E - V_{\text{eff}}(r))}}. \tag{3.32}$$

The angle of advance over a complete cycle is twice this.

Now, *if* the orbit is going to close on itself, there has got to be some number of radial periods during which the angle of advance of θ is a multiple of 2π. In other words,

$$\text{Closure of the orbit} \quad \Leftrightarrow \quad \Phi/2\pi \text{ is a rational number.}$$

In general, we've no right to expect that condition to hold. It's really remarkable that it is true for an inverse square force, and it's to that problem that we now turn attention.

Exercise Draw some closed and non-closed orbits.

That is as far as we can go with a generic central force. But it's a considerable reduction! We have found that the motion is actually confined to a two-dimensional surface in our original twelve-dimensional phase space.

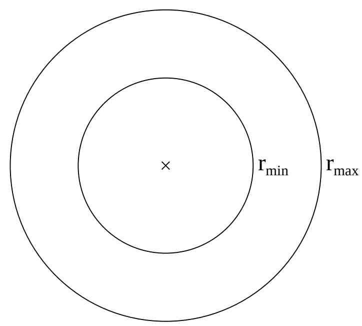

Figure 4.6: The effective potential determines a minimum and maximum radius, depending upon E and ℓ, between which the orbit must remain.

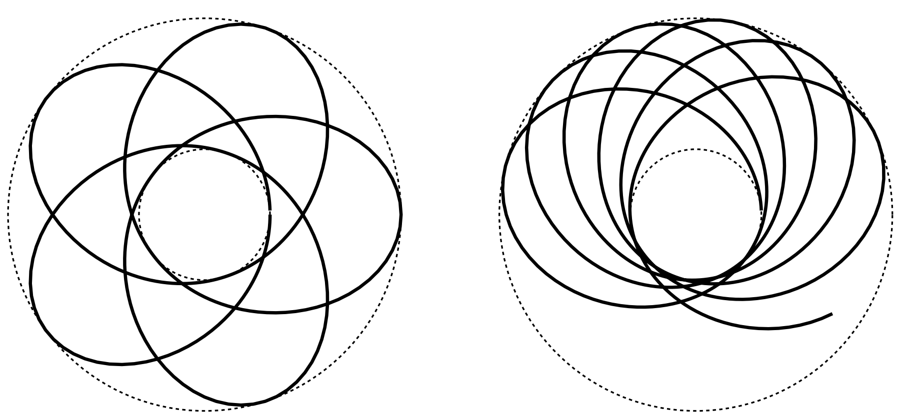

Figure 4.7: The orbit on the left closes after four periods of θ. The one on the right looks as though it may never do so.

4.4 The Kepler Problem

4.4.1 Kepler's Laws: historical prelude

In 1601 Johannes Kepler set about the task of analyzing the orbit of Mars. He was armed with the logbooks of Tycho Brahe, containing by far the most accurate and painstaking measurements of planetary postions achieved up to that time. Would he even have started if he had known it would take five years? In 1609, *Astronomia Nova* was published, and in it were statements of what have since become known as Kepler's first and second laws of planetary motion. The third followed some years later. Here are

Kepler's Laws of Planetary Motion

1. All planetary orbits are ellipses, with the Sun at one focus.

2. The vector joining the Sun to a planet sweeps out equal areas in equal times.

3. The square of the period of a planet is proportional to the cube of its mean distance to the sun.

Three quarters of a century later, in 1684, notions about gravity and possible forms for its dependence on distance were starting to form in the minds of several people. Sir Christopher Wren challenged Robert Hooke and Edmond Halley to prove that an inverse-square force law would lead to Kepler's first law, and he offered a small prize (worth 40 shillings!) to the one who could do it first. Halley happened to mention the problem to Newton. Here is DeMoivre's account of the encounter.

> After they had been some time together, the Dr. [Halley] asked him what he thought the curve would be that would be described by the planets supposing the force of attraction towards the sun to be reciprocal to the square of their distance from it. Sir Isaac replied immediately that it would be an ellipsis. The Doctor, struck with joy and amazement, asked him how he knew it. Why, saith he, I have calculated it. Whereupon Dr. Halley asked him for his calculation without any further delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it and then to send it him. Sir Isaac, in order to make good his promise, fell to work again, but he could not come to that conclusion which he thought he had before examined with care. However, he attempted a new way which, though longer than the first,

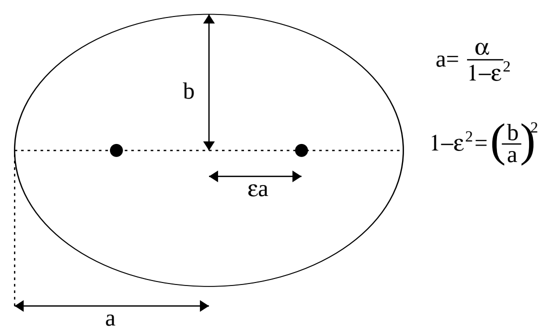

Figure 4.8: Some standard notation for ellipses. The semimajor axis (i.e., half the long axis) is a and the semiminor axis is b. The heavy dots indicate the foci. The distance from the center of the ellipse to either focus is ϵa, where ϵ is the **eccentricity**.

> brought him again to his former conclusion. Then he examined carefully what might be the reason why the calculation he had undertaken before did not prove right, and he found that, having drawn an ellipsis coarsely with his own hand, he had drawn the two axes of the curve, instead of drawing two diameters somewhat inclined to one another, whereby he might have fixed his imagination to any two conjugate diameters, which was requisite he should do. That being perceived, he made both his calculations agree together.

Three years later, in 1687, Newton's *Principia Mathematica* was published. High time, as he'd been sitting on many of the results for nearly twenty years. It set forth the basic principles of mechanics, the solution of the Kepler problem and the foundations of modern celestial mechanics, and an excellent recipe for chocolate cake. Unfortunately it was written in latin so it never made it onto the bestseller list.

4.4.2 finding the orbits: the $u = 1/r$ trick

The inverse-square central force, of which gravity is a prime example, is a very special one. In this section, the problem of motion in such a potential is completely solved. The potential for such a force is

$$V(r) = -\frac{k}{r}. \tag{4.33}$$

The sign is chosen so that $k > 0$ corresponds to an attractive interaction, the case usually of interest. For gravity, $k = GMm$. Inserting this into the earlier result, Eq. (3.25) for the energy,

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r}. \tag{4.34}$$

Everything we've done up to now has been very systematic. Further progress relies on a trick. For no clear reason, we make the substitution

$$u = \frac{1}{r}. \tag{4.35}$$

EOM of r:

Using this, we rewrite the expression for E with u and θ instead of r and t. That requires the relations

$$\begin{aligned} \dot{r} &= -\frac{\dot{u}}{u^2} \\ \dot{u} &= \frac{du}{d\theta}\dot{\theta} \\ \dot{\theta} &= \frac{\ell}{\mu}u^2 \end{aligned} \tag{4.36}$$

with the result

$$E = \frac{\ell^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku. \tag{4.37}$$

Exercise Check all those things. It's straightforward algebra.

It's not yet clear what has been gained by this. But E is a constant of the motion, so

$$\frac{dE}{d\theta} = 0, \tag{4.38}$$

which implies

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu k}{\ell^2} = 0. \tag{4.39}$$

Now we have something. This is the equation for a an undamped simple harmonic oscillator with constant driving! We studied those to death in the last chapter, so the solution is very easy. With the abbreviation

$$\alpha = \frac{\ell^2}{\mu k}, \tag{4.40}$$

and choosing the direction $\theta = 0$ right, we can write

$$u = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right). \tag{4.41}$$

The constant of integration ϵ needs to be related to physical properties of the current problem. ϵ is found by substituting back into the expression for the energy, with the result

$$1 - \epsilon^2 = -\frac{2E\ell^2}{\mu k^2}. \tag{4.42}$$

Kepler 1:

Finally, putting it back in terms of r,

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta. \tag{4.43}$$

Though you may not recognize it, this is the formula for an ellipse of eccentricity ϵ with one focus at the origin. The amazing thing is that the orbits are closed! We have derived Kepler's first law.

Kepler 2:

Kepler's 2nd law holds for any central force, and we could have proved it in the previous section. If the planet is at a radius of r and the angle θ changes by a teensy bit $\delta\theta$, the area swept out is a triangle with a base of $r\,\delta\theta$ and an altitude of r, giving an area $(r\,\delta\theta)(r)/2$. (See figure 4.9.) Dividing this by the time $\delta\theta/\dot{\theta}$ required for that motion,

$$\frac{d}{dt}\text{Area} = \frac{r^2\dot{\theta}}{2} = \frac{\ell}{\mu}. \tag{4.44}$$

Kepler's 2nd Law is therefore a consequence of conservation of angular momentum, which holds for any central force.

Kepler 3:

A little bit of algebra, using Figure 4.8 gets a formula for the semimajor axis of the elliptical orbit in terms of energy and force constant k:

$$a = \text{semimajor axis} = -\frac{k}{2E}. \tag{4.45}$$

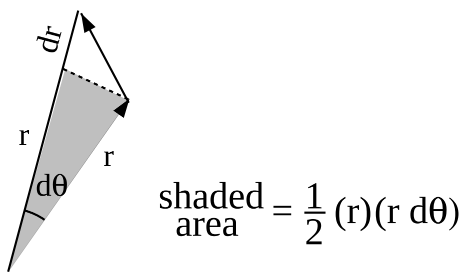

Figure 4.9: Area is swept out by the radius vector at a constant rate for any central force. The area of the unshaded triangle can be thrown away because it is second order in differentials, being proportional to $dr\, d\theta$.

Using the formula in the figure to get the semiminor axis and the fact that an ellipse has area πab, and a bit more scratch paper,

$$\text{Area} = \frac{\pi \ell k}{\sqrt{8|E|^3 \mu}}. \tag{4.46}$$

Oops. We want to relate period to mean distance, but we've got a relation between area and energy. I'll leave it to you.

$\boxed{\textsf{Exercise}}$ Finish proving Kepler's 3rd Law. Change area for semimajor axis, and area for period (K2). Also notice that mean distance is proportional to semimajor axis though you may not know the proportionality.

4.5 Notes